

CURRENT AND MEASURE HOMOLOGIES

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ABSTRACT. The space of all normal currents with compact support in a metric space in the sense of Ambrosio and Kirchheim becomes a chain complex. Its homology is proved to be naturally isomorphic to the measure homology of Thurston on the category of all weakly locally Lipschitz contractible metric spaces and all locally Lipschitz maps. The class of weakly locally Lipschitz contractible metric spaces is large, including all finite dimensional Alexandrov spaces of curvature bounded locally from below or all locally CAT spaces.

To prove this result, a topology on a space of Lipschitz maps between arbitrary metric spaces is introduced and its fundamental properties are investigated. Further, the C^1 -topology on the space of all C^1 -maps between Riemannian manifolds is characterized by our topology.

1. INTRODUCTION

Currents in smooth manifolds were introduced by de Rham ([D]), which were defined to be continuous linear functionals on the spaces of smooth forms. Currents in metric spaces (called metric currents) were introduced by Ambrosio and Kirchheim ([AK]), by providing a space of formal forms on metric spaces. When we will say just currents, they will indicate metric currents. The space of all normal currents of compact support in a metric space (and in a smooth manifold) is known to be a chain complex. We call its homology the *current homology*. In the present paper, we give an answer what current homology is, on a large class of metric spaces. This is given by the *measure homology*. Here, the measure homology, which is also called the Milnor-Thurston homology, was introduced by Thurston ([T]) to compute the simplicial volume of hyperbolic manifolds. We will review its precise definition in §3.

To state our results, let us fix notation. For a metric space X , we denote by $\mathbf{N}_*^c(X)$ and $\mathcal{C}_*(X)$ the chain complex of normal currents with compact support and the measure chain complex, respectively. Their homologies are denoted by $\mathbf{H}_*(X)$ and $\mathcal{H}_*(X)$, respectively. Thurston originally defined the measure chain complex for smooth manifolds. It will be denoted by $\mathcal{C}_*^{\text{diff}}(X)$ and called the smooth (or original) measure chain complex. In our paper, we employ a slightly modified chain

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complex which can be defined for any topological space, considered by Hansen ([H]) and Zastrow ([Z]), as the measure chain complex.

Let us recall some notion for metric spaces.

Definition 1.1 ([Mi]). A metric space X is said to be *weakly locally Lipschitz contractible*, shortly WLLC, if for every $x \in X$ and open set O of X , there exist an open set O' of X with $x \in O' \subset O$, a Lipschitz map $h : O' \times [0, 1] \rightarrow O$ and a point $y \in O$ such that h_0 is the inclusion $O' \hookrightarrow O$ and h_1 is the constant map valued y , where h_t denotes the map $h(\cdot, t)$ for $t \in [0, 1]$.

Example 1.2. There are many objects being weakly locally Lipschitz contractible in metric geometry: Normed spaces, Riemannian manifolds, locally CAT-spaces, and finite dimensional locally Alexandrov spaces, satisfy this property. Here, locally CAT spaces (resp. locally Alexandrov spaces) are length metric spaces of curvature bounded locally from above (resp. below) in the sense of Alexandrov. For their precise definitions and fundamental properties, we refer to [BGP] and [BBI]. Among these examples, it is not trivial that finite dimensional locally Alexandrov spaces are WLLC. It was proved in [MY]. Further, these spaces were proved to satisfy a stronger condition called the strong local Lipschitz contractibility, there. For other properties similar to WLLC and their relation, we refer [Mi] and its reference.

Further, by the definition, for a WLLC metric space X , if O is an open subset of it, then O is also WLLC, and if another metric space Y is locally bi-Lipschitz homeomorphic to X , then Y is WLLC.

Based on the above preliminaries, a main result is stated as follows.

Theorem 1.3. *On the category of all WLLC metric spaces and all locally Lipschitz maps, there is a natural isomorphism $\mathbf{H}_* \cong \mathcal{H}_*$ between the current and measure homologies.*

A more precise statement of this will be in §1.1.

The following statements follows from the definitions and fundamental properties of currents and the measure homology together with Theorem 1.3.

Corollary 1.4. *Let X and Y be two weakly locally Lipschitz contractible metric spaces. If they are homotopic to each other, then their current homologies $\mathbf{H}_*(X)$ and $\mathbf{H}_*(Y)$ are isomorphic to each other.*

Corollary 1.5. *Let X be a weakly locally Lipschitz contractible metric space of Hausdorff dimension less than n , for $n \in \mathbb{Z}_{\geq 0}$. Then, its measure homology $\mathcal{H}_k(X)$ is trivial for every $k \geq n$.*

Corollary 1.6. *Let X be a WLLC metric space. Suppose that X is homotopic to a finite CW-complex. Then, $\mathbf{H}_k(X)$ are finite dimensional for all $k \geq 0$.*

We will remark that a relative version of Theorem 1.3 also holds. Using that, we obtain

Corollary 1.7. *Let X be a finite-dimensional locally Alexandrov space, and E a possibly empty subset of X . Assume that E is either closed intrinsic, open, or extremal. Then, there is a natural isomorphism*

$$\mathcal{H}_*(X, E) \cong \mathbf{H}_*(X, E)$$

between functors \mathcal{H}_ and \mathbf{H}_* on the category of such pairs (X, E) and locally Lipschitz maps.*

Here, a subset E in an Alexandrov space X is said to be intrinsic if the restricted metric of X to E is intrinsic. For the definition of subsets being extremal, we refer to [PP] and [Pet]. We just remark that a subset E of X as in Corollary 1.7 is also strongly locally Lipschitz contractible in the sense of [MY].

1.1. More precise statement. Let us first give an intuitive explanation of a coincidence of currents and measures in the 0-th chain groups. After that, we state a precise version of Theorem 1.3.

By the definition, 0-currents are continuous functionals on the space of formal 0-forms (which are just real-valued bounded Lipschitz functions) on a metric space X . They are like Schwartz's distributions. On the other hands, a signed Borel measure on X can be actually regarded as a distribution, by the integration of functions with respect to the measure. Thus, we have an intuitive identification:

$$(1.1) \quad \{0\text{-currents in } X\} = \{\text{signed Borel measures on } X\}.$$

We will check that the above equality actually has the meaning (Lemma 3.20).

In the present paper, we consider a higher dimensional analogue of this correspondence (1.1). We will provide a way to regard a signed measure on the space of singular Lipschitz simplices in a metric space X as a current in X . Further, we provide some chain complex $\mathcal{C}_*^L(X)$ with a chain map

$$(1.2) \quad \mathcal{C}_*^L(X) \rightarrow \mathcal{C}_*(X)$$

and define a chain map

$$(1.3) \quad \mathcal{C}_*^L(X) \rightarrow \mathbf{N}_*^c(X).$$

The homology of $\mathcal{C}_*^L(X)$ is denoted by $\mathcal{H}_*^L(X)$.

More precisely, the k -th complex group $\mathcal{C}_k^L(X)$ is given by some space consisting of signed "Borel" measures on a "topologized" space $\text{Lip}(\Delta^k, X)$ of all singular Lipschitz k -simplices in X . Here, we should give a suitable topology on the space of Lipschitz maps to guarantee that the maps (1.2) and (1.3) are well posed. Its topology will be given to have several reasonable properties (Theorem 3.1 and §5). We will remark that the map (1.3) is ill-defined if the space of singular Lipschitz

simplices is merely equipped with the compact-open topology (Remark 3.17).

Theorem 1.3 can be stated more precisely as follow.

Theorem 1.8. *On the category of all metric spaces and all locally Lipschitz maps, the correspondence (1.2) and (1.3) are natural transformations between covariant functors \mathcal{C}_* , \mathcal{C}_*^L and \mathbf{N}_* to the category of all chain complexes of real vector spaces and all chain maps.*

Further, if the natural transformations (1.2) and (1.3) are restricted to the full subcategory consisting of all weakly locally Lipschitz contractible metric spaces, then the corresponding induced maps between homologies

$$\mathcal{H}_* \leftarrow \mathcal{H}_*^L \rightarrow \mathbf{H}_*$$

are isomorphic.

In the rest of this section, we review known results related to our result, and remark their relations.

1.2. Remarks on smooth case. It was proved that the space of normal currents in the classical sense is identified with the space of normal metric currents ([AK]). Namely, for a smooth complete Riemannian manifold M , we can use the same symbol $\mathbf{N}_*^c(M)$ to indicate the space of compactly supported normal currents in the both senses of de Rham and Ambrosio-Kirchheim.

Thurston's original measure k -th chain group $\mathcal{C}_k^{\text{diff}}(M)$ of a smooth manifold M , is defined to be some space consisting of signed Borel measures on the space $C^1(\Delta^k, M)$ of all singular C^1 -simplices equipped with the C^1 -topology. Using a terminology in §3, it is defined as $\mathcal{C}_k^{\text{diff}}(M) = \mathcal{M}_c(C^1(\Delta^k, M))$. From the definition, there is a canonical chain map $\mathcal{C}_*^{\text{diff}}(M) \rightarrow \mathcal{C}_*(M)$.

It is considered that our Lipschitz measure chain complex \mathcal{C}_*^L on a category of metric spaces corresponds to the original measure complex $\mathcal{C}_*^{\text{diff}}$ on the category of smooth manifolds. Further, a way to construct a current in M from a measure $\mu \in \mathcal{C}_*^{\text{diff}}(M)$, was remarked by Thurston in [T, §6.1]. It actually gives a map

$$(1.4) \quad \mathcal{C}_*^{\text{diff}}(M) \rightarrow \mathbf{N}_*^c(M),$$

which corresponds to our map (1.3). This map is easily checked to be a chain map from the definition (or an argument similar to the proof of Theorem 3.19.) However, he did not mention that this map induces an isomorphism between homologies.

In [Ra, §11], Ratcliffe considered the *algebraic* dual $\mathcal{D}_*(M)$ of the space $\Omega^*(M)$ of smooth forms for a smooth manifold M , and explicitly defined a chain map

$$\mathcal{C}_*^{\text{diff}}(M) \ni \mu \mapsto f_\mu \in \mathcal{D}_*(M).$$

There, the functional f_μ on $\Omega^*(M)$ is not checked to be a current. However, this map is no other than the map (1.4). Namely, the image of it is contained in $\mathbf{N}_*^c(X)$. He proved that the chain map $\mathcal{C}_*^{\text{diff}}(M) \rightarrow \mathcal{D}_*(M)$ induces a surjective linear map between homologies, by using de Rham's theorem ([Ra, Theorem 11.7.1]).

Theorem 1.8 will be proved by using a quite different way to the proof of [Ra, Theorem 11.7.1]. We will use cosheaf theory introduced by Bredon ([B]) to prove Theorem 1.8. Since our proof of Theorem 1.8 works for smooth category, we obtain

Theorem 1.9. *On the category of all smooth manifolds and all smooth maps, the natural transformations*

$$\mathcal{C}_* \leftarrow \mathcal{C}_*^{\text{diff}} \rightarrow \mathbf{N}_*^c$$

induce isomorphisms between homologies. In particular, the homology of compactly supported normal currents in the original sense of de Rham is naturally isomorphic to the original measure homology of Thurston.

Since all manifolds are CW-complexes, Theorem 1.9 also follows from results in [H], [Z] and [Mo] that will be stated in §1.4. So, we omit the proof of Theorem 1.9 in the paper.

1.3. Compatibility. Our natural transformation $\mathcal{C}_*^L \rightarrow \mathbf{N}_*^c$ (consisting of chain maps) as in (1.3) is compatible to other known chain maps in the following sense. Let C_* denotes the functor of usual singular chain complex of real coefficient. There is a natural transformation consisting of chain maps

$$(1.5) \quad \delta : C_* \rightarrow \mathcal{C}_*$$

on the category of all Hausdorff spaces and all continuous maps. Here, $\delta : C_*(X) \rightarrow \mathcal{C}_*(X)$ is defined by the linear extension of the map sending each singular simplex to its Dirac delta measure, for every Hausdorff space X . For a metric space Y , let us denote by $C_*^L(Y)$ the Lipschitz singular chain complex which is a subcomplex of $C_*(Y)$ generated by Lipschitz singular simplices. A map $\delta : C_*^L \rightarrow \mathcal{C}_*^L$ corresponding to (1.5) is defined on the category of all metric spaces and all locally Lipschitz maps. Riedweg and Schäppi ([RS]) considered a natural transformation consisting of chain maps

$$(1.6) \quad [\cdot] : C_*^L \rightarrow \mathbf{N}_*^c$$

on the category of metric spaces. We will review the definition of it in §3.5 of this paper. Our maps $\mathcal{C}_* \leftarrow \mathcal{C}_*^L \rightarrow \mathbf{N}_*^c$ are compatible to them in the sense that the following diagram consisting of natural

transformations commutes:

$$(1.7) \quad \begin{array}{ccc} C_* & \longleftarrow & C_*^L \\ \delta \downarrow & & \downarrow \delta \\ \mathcal{C}_* & \xleftarrow{(1.2)} & \mathcal{C}_*^L \end{array} \quad \begin{array}{c} \searrow [\cdot] \\ \longrightarrow \mathbf{N}_*^c \end{array} \quad \begin{array}{c} \xrightarrow{(1.3)} \end{array}$$

on the category of all metric spaces and all locally Lipschitz maps.

1.4. Correspondence to singular homology. In this subsection, we review and state relations between the usual real singular, the current and the measure homologies, on several categories.

First, we recall the following

Theorem 1.10 ([Y], [Mi]). *On the category of all pairs of weakly locally Lipschitz contractible metric spaces and all locally Lipschitz maps, the inclusion $C_*^L \rightarrow C_*$ induces an isomorphism between homologies. Here, a pair (X, A) of metric spaces is said to be weakly locally Lipschitz contractible if so are both X and A , where A is a subset of X endowed with the restricted metric.*

For more details, Yamaguchi introduced the notion of being locally Lipschitz contractible for metric spaces and proved Theorem 1.10 for such spaces ([Y]). The author extended it for weakly locally Lipschitz contractible spaces ([Mi]). Further, the coincidence of singular and singular Lipschitz homologies holds for every coefficient on the category of WLLC spaces.

The relative current homology functor \mathbf{H}_* was proved to satisfy Eilenberg-Steenrod axiom of homology theory of real coefficient on the category of all pairs of metric spaces and all locally Lipschitz maps by Mongodi ([Mo, Theorem 8]). There, he called a pair of metric spaces a pair of a locally compact metric space and its closed subset. He considered a (locally compact) metric space which is both locally Lipschitz contractible and has the topological type of a CW-complex. We call such a space an LLC CW-complex. He proved that for any continuous map $f : X \rightarrow Y$ between such spaces X and Y , there is a homotopy from f to a locally Lipschitz map $f' : X \rightarrow Y$ ([Mo, Lemma 7]). Hence, for a homotopy class represented by a continuous map $f : X \rightarrow Y$ with $f(A) \subset B$, where $A \subset X$ and $B \subset Y$ are LLC CW-subcomplexes, we can define induced maps $f_* : H_*^L(X, A) \rightarrow H_*^L(Y, B)$ and $f_* : \mathbf{H}_*(X, A) \rightarrow \mathbf{H}_*(Y, B)$. Here, $H_*^L(X, A)$ denotes the homology of $C_*^L(X)/C_*^L(A)$. By summarizing above and using the uniqueness of homology theories, he concluded

Theorem 1.11 ([Mo, Corollary 9]). *On the category of all pairs of LLC CW-complexes and all continuous maps, the natural transformations $\mathbf{N}_*^c \leftarrow C_*^L \rightarrow C_*$ induce isomorphisms between homologies.*

The relative measure homology functor \mathcal{H}_* was proved to satisfy Eilenberg-Steenrod axiom due to Hansen [H] and Zastrow [Z], independently, on the category of all pairs of metrizable spaces and all continuous maps and on the category of all pairs of CW-complexes and all continuous maps. Further, by using the uniqueness of homology theories, they proved

Theorem 1.12 ([H], [Z]). *The natural chain map (1.5) induces an isomorphism between homologies on the category of all CW-pairs and all continuous maps.*

On the category of all smooth manifolds and all smooth maps, the natural transformations

$$\begin{array}{ccc} C_*^{\text{diff}} & \longrightarrow & C_* \\ \downarrow & & \downarrow \\ \mathcal{C}_*^{\text{diff}} & \longrightarrow & \mathcal{C}_* \end{array}$$

induce isomorphisms between homologies. Here, $C_^{\text{diff}}(M)$ denotes the subcomplex of $C_*(M)$ generated by smooth singular simplices in a smooth manifold M , and $C_*^{\text{diff}} \rightarrow \mathcal{C}_*^{\text{diff}}$ is a canonical map similar to $C_* \rightarrow \mathcal{C}_*$.*

As seen in the proofs of Theorems 1.11 and 1.12, the uniqueness of homology theories is effective to show a coincidence of ordinary homology theories on a category consisting of topological spaces which have the homotopy type of a CW-complex. In §6 of the present paper, we will give an example of a WLLC metric space which has no homotopy type of a CW-complex. Therefore, the proof of Theorem 1.8 is more involved, and needs another argument.

Due to Theorems 1.8 and 1.12, we obtain

Corollary 1.13. *On the category of all WLLC metric spaces which are homotopic to CW-pairs and all locally Lipschitz maps, the natural transformations (1.7) induce isomorphisms between homologies. In particular, the current homology \mathbf{H}_* and the singular homology H_* of real coefficient are naturally isomorphic on this category.*

For instance, if a Hausdorff space is countable-dimensional and locally contractible, then it is homotopic to a CW-complex. Here, for a metric space, it is countable-dimensional if it is a countable union of 0-dimensional subspaces. Finite topological dimensional metric spaces are countable-dimensional.

Corollary 1.14. *Let (X, E) be as in Corollary 1.7. Then, there is a natural isomorphism between the current homology $\mathbf{H}_*(X, E)$ and the real singular homology $H_*(X, E)$.*

The organization of this paper is as follows. In §2, we recall the definition of metric currents in the sense of Ambrosio and Kirchheim and

define the current homology of metric spaces. In §3, we recall the definition of the measure homology. We state some result in a topology on the set of all bounded Lipschitz maps between arbitrary metric spaces (Theorem 3.1). After that, we define some chain complex, that will be called the Lipschitz measure homology of a metric space, and define chain maps from it to the measure and current homologies as (1.2) and (1.3) by using our topology. There, we remark why we introduce a new topology on the space of Lipschitz maps instead of the compact-open topology. In §4, we recall the notion of cosheaf and its fundamental property, and prove Theorem 1.8 (and Theorem 1.3) by using the cosheaf theory. In §5, we prove Theorem 3.1, i.e., we give a reasonable topology on the set of all bounded Lipschitz maps between metric spaces. Further, we investigate additional properties of our topology. In §6, we give an example of a weakly locally Lipschitz contractible metric space which has no homotopy type of a CW-complex.

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2. CURRENTS AND ITS HOMOLOGIES

Let us recall the definition of metric currents in the sense of Ambrosio and Kirchheim ([AK]).

Let X denote a metric space. Let $\text{Lip}(X)$ be the set of all Lipschitz functions from X to \mathbb{R} and $\text{Lip}_b(X)$ the subset of $\text{Lip}(X)$ consisting of bounded functions. For a map f between metric spaces, we denote by $\text{Lip}(f)$ its Lipschitz constant. Let k denote a nonnegative integer. The space $\mathcal{D}^k(X) := \text{Lip}_b(X) \times (\text{Lip}(X))^k$ is considered as the space of k -forms on X . An element $(f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(X)$ is written as $fd\pi_1 \wedge \dots \wedge d\pi_k$ or $fd\pi$ for shortly.

Definition 2.1 ([AK]). A k -current in X is a multilinear map

$$T : \mathcal{D}^k(X) \rightarrow \mathbb{R}$$

such that it satisfies the locality, is continuous and has finite mass in the following sense. Here, T satisfies the locality if $T(fd\pi) = 0$ whenever π_i is constant on $\{f \neq 0\}$ for some i ; T is continuous if

$$T(fd\pi) = \lim_{h \rightarrow \infty} T(fd\pi^h)$$

whenever a sequence $\pi^h = (\pi_i^h) \in (\text{Lip}(X))^k$, $h \in \mathbb{N}$, converges to $\pi = (\pi_i)$ pointwise as $h \rightarrow \infty$ with $\sup_{i,h} \text{Lip}(\pi_i^h) < \infty$; and T has

finite mass if there is a finite tight Borel measure μ on X satisfying

$$|T(fd\pi)| \leq \prod_{i=1}^k \text{Lip}(\pi_i) \int_X |f| d\mu$$

for all $fd\pi \in \mathcal{D}^k(X)$.

Let T denote a k -current in X . The *support* of T is defined by the intersection of all supports of μ satisfying the finite mass axiom for T . We deal with only currents of compact support in the paper. The *boundary* of T is a multi-linear map $\partial T : \mathcal{D}^{k-1}(X) \rightarrow \mathbb{R}$ defined by

$$\partial T(fd\pi) = T(df \wedge d\pi).$$

By the locality, $\partial\partial T = 0$ holds. The boundary ∂T satisfies the continuity and locality. If ∂T has finite mass, then T is said to be *normal*. The set of all normal k -currents in X is denoted by $\mathbf{N}_k(X)$. Thus, $\mathbf{N}_*(X)$ becomes a chain complex. Since supports being compact preserves under the boundary, the space $\mathbf{N}_*^c(X)$ of all compactly supported normal currents is also a chain complex. In the present paper, we consider its homology, denoted by

$$\mathbf{H}_*(X) := H_*(\mathbf{N}_*^c(X)).$$

We call it the *current homology*. For the empty-set, we set $\mathbf{N}_*^c(\emptyset) = 0$.

For another metric space Y with a locally Lipschitz map $\phi : X \rightarrow Y$, we have a chain map

$$\phi_\# : \mathbf{N}_*^c(X) \rightarrow \mathbf{N}_*^c(Y)$$

defined by

$$\phi_\# T(fd\pi) = T(f \circ \phi d(\pi \circ \phi))$$

for all $T \in \mathbf{N}_k^c(X)$, $fd\pi \in \mathcal{D}^k(X)$ and $k \geq 0$. This is actually defined, since normal currents are compactly supported (see [Mi]). Thus, the chain complex \mathbf{N}_*^c is a covariant functor from the category of all metric spaces and all locally Lipschitz maps to the category of all chain complexes and all chain maps. The current homology \mathbf{H}_* is a covariant functor to the category of all vector spaces and all linear maps.

For a metric space X and its subset A , we say that (X, A) is a pair of metric spaces. Since the inclusion $A \hookrightarrow X$ induces an injective chain map $\mathbf{N}_*^c(A) \rightarrow \mathbf{N}_*^c(X)$, we can regard $\mathbf{N}_*^c(A)$ as a subcomplex of $\mathbf{N}_*^c(X)$. We set

$$\mathbf{N}_*^c(X, A) := \mathbf{N}_*^c(X) / \mathbf{N}_*^c(A).$$

Its homology is denoted by

$$\mathbf{H}_*(X, A)$$

called the current homology of (X, A) . A map $f : (X, A) \rightarrow (Y, B)$ between pairs of metric spaces is said to be locally Lipschitz if so is $f : X \rightarrow Y$. If $f : (X, A) \rightarrow (Y, B)$ is a locally Lipschitz map, then a chain

map $f_{\#} : \mathbf{N}_*^c(X, A) \rightarrow \mathbf{N}_*^c(Y, B)$ and a linear map $f_* : \mathbf{H}_*(X, A) \rightarrow \mathbf{H}_*(Y, B)$ are induced.

3. (LIPSCHITZ) MEASURE HOMOLOGY

In this section, we recall the definition of measure chain complex of topological spaces. We introduce some chain complex for metric spaces with a chain map from it to the measure chain complex. It will be called the Lipschitz measure chain complex. To achieve our purpose, we will equip a reasonable topology on the set of all bounded Lipschitz maps between metric spaces. Finally, we provide a natural chain map from the Lipschitz measure chain complex to the complex of compactly supported normal currents. Our topology on the space of bounded Lipschitz maps guarantees that these chain maps are well posed.

3.1. Fixing terminology from measure theory. Before defining the measure homology, let us fix the terminology and notation about measures. Let (T, \mathcal{A}) be a measurable space. We say that a function $\mu : \mathcal{A} \rightarrow \mathbb{R} \cup \{\infty, -\infty\}$ is a signed measure if $\mu(\emptyset) = 0$, the image of it does not contains both values ∞ and $-\infty$, and it is σ -additive. A subset D of T which is not necessarily measurable is called a *determination set* of a signed measure μ on (T, \mathcal{A}) if every measurable set $A \in \mathcal{A}$ contained in $T - D$ is of zero measure in μ .

Let S be a topological space. Let us denote by $\mathcal{M}_c(S)$ the real vector space of all signed Borel measures on S of finite total variation having a compact determination set. For any continuous map $f : S \rightarrow S'$ between topological spaces, a linear map $f_{\#} : \mathcal{M}_c(S) \rightarrow \mathcal{M}_c(S')$ is given by sending $\mu \in \mathcal{M}_c(S)$ to the push-forward measure $f_{\#}\mu = \mu(f^{-1}(\cdot)) \in \mathcal{M}_c(S')$. Obviously, if $f : S \rightarrow S'$ is a topological embedding, then $f_{\#}$ is injective.

For a Borel set A of S and a signed Borel measure μ on S , we define a signed Borel measure $\mu|_A$ on S by

$$\mu|_A(B) = \mu(A \cap B)$$

for every Borel set B of S . By the definition, it has a determination set A . Further, we use the same symbol $\mu|_A$ meaning the restriction of μ to the Borel σ -algebra of A , which is a signed Borel measure on A .

3.2. Measure homology. In this subsection, let X denote a topological space. For $k \geq 0$, we denote by Δ^k a regular k -simplex, where Δ^0 denotes a one-point set. Let us denote by $C(\Delta^k, X)$ the space of all singular k -simplices in X with the compact-open topology. Note that X is Hausdorff if and only if so is $C(\Delta^k, X)$. Recall that if X is a metric space, then the compact-open topology on $C(\Delta^k, X)$ coincides with the uniform topology.

The measure k -th chain group of X is defined by

$$\mathcal{C}_k(X) := \mathcal{M}_c(C(\Delta^k, X)).$$

For $i = 0, \dots, k$, the restriction $r_i : C(\Delta^k, X) \rightarrow C(\Delta^{k-1}, X)$ to the i -th face of Δ^k is continuous in the compact-open topology. This induces a linear map $r_{i\#} : \mathcal{C}_k(X) \rightarrow \mathcal{C}_{k-1}(X)$. Then, the following map given by

$$\partial = \sum_{i=0}^k (-1)^i r_{i\#} : \mathcal{C}_k(X) \rightarrow \mathcal{C}_{k-1}(X)$$

is easily checked to satisfy $\partial\partial = 0$. So, $(\mathcal{C}_*(X), \partial)$ becomes a chain complex and is called the *measure chain complex*. Its homology is called the *measure homology* and is denoted by

$$\mathcal{H}_*(X).$$

Let Y denote another topological space with $\phi : X \rightarrow Y$ a continuous map. Since the composition $\phi_{\#} : C(\Delta^k, X) \rightarrow C(\Delta^k, Y); \sigma \mapsto \phi \circ \sigma$ is continuous, it induces a chain map $\phi_{\#} : \mathcal{C}_*(X) \rightarrow \mathcal{C}_*(Y)$ by the push-forward of measures.

Let A be a subspace of X . We say that such an (X, A) is a topological pair. The inclusion $A \hookrightarrow X$ induces a topological embedding $C(\Delta^k, A) \rightarrow C(\Delta^k, X)$ for every $k \geq 0$. Hence, it induces an injective chain map $\mathcal{C}_*(A) \rightarrow \mathcal{C}_*(X)$. Thus, we regard $\mathcal{C}_*(A)$ as a subcomplex of $\mathcal{C}_*(X)$. The quotient $\mathcal{C}_*(X)/\mathcal{C}_*(A)$ is denoted by

$$\mathcal{C}_*(X, A)$$

called the measure chain complex of (X, A) . Its homology is denoted by

$$\mathcal{H}_*(X, A)$$

called the measure homology of the pair (X, A) . When $A = \emptyset$, we identify $\mathcal{H}_*(X, \emptyset)$ with $\mathcal{H}_*(X)$.

3.3. A topology on spaces of bounded Lipschitz maps. In this subsection, let X, Y, Z and W denote metric spaces. We denote by $\text{Lip}(Z, X)$ (resp. $\text{Lip}_b(Z, X)$) the set of all Lipschitz (resp. bounded Lipschitz) maps from Z to X . We now equip $\text{Lip}_b(Z, X)$ with some reasonable topology. One of main results in the present paper is the following

Theorem 3.1. *For any metric spaces Z and X , there is a topology on $\text{Lip}_b(Z, X)$ satisfying the following.*

- (0) *The topology on $\text{Lip}_b(Z, X)$ is metrizable and finer than the topology induced by the supremum distance.*
- (1) *If a sequence f_j converges to f in the topology on $\text{Lip}_b(Z, X)$, then $\sup_j \text{Lip}(f_j) < \infty$.*

- (2) Let $\phi : X \rightarrow Y$ be a Lipschitz map between metric spaces. Then, a map $\phi_{\#} : \text{Lip}_b(Z, X) \rightarrow \text{Lip}_b(Z, Y)$ given by $\sigma \mapsto \phi \circ \sigma$ is continuous. Further, if X is a subset of Y and ϕ is the inclusion $X \hookrightarrow Y$, then $\phi_{\#}$ is a topological embedding.
- (3) Let $\psi : W \rightarrow Z$ be a Lipschitz map between metric spaces. Then, a map $\psi^{\#} : \text{Lip}_b(Z, X) \rightarrow \text{Lip}_b(W, X)$ given by $\sigma \mapsto \sigma \circ \psi$ is continuous.
- (4) A canonical map

$$\text{Lip}_b(Z, X) \times \text{Lip}_b(W, Y) \rightarrow \text{Lip}_b(Z \times W, X \times Y)$$

defined by $(\phi, \psi) \mapsto \phi \times \psi$ is continuous, where $\phi \times \psi$ is given by $(\phi \times \psi)(z, w) = (\phi(z), \psi(w))$ for $(\phi, \psi) \in \text{Lip}_b(Z, X) \times \text{Lip}_b(W, Y)$ and $(z, w) \in Z \times W$.

- (5) Let $\{*\}$ denote a single-point set. Then, a canonical map

$$\text{Lip}_b(\{*\}, X) \ni f \mapsto f(*) \in X$$

is homeomorphic.

Theorem 3.1 will be proved in §5. We mainly deal with compact metric spaces as domains of Lipschitz maps later. For a compact metric space Z , $\text{Lip}_b(Z, X) = \text{Lip}(Z, X)$ as sets. In this case, we always consider that $\text{Lip}(Z, X)$ has the topology satisfying Theorem 3.1.

Remark 3.2. Among properties of a topology as in Theorem 3.1, the property (1) is a crucial difference from the compact-open topology. Indeed, there is a sequence of real-valued Lipschitz functions on $[0, 1]$ converging to a Lipschitz function uniformly such that the Lipschitz constants diverges to infinity. Hence, any topology satisfying Theorem 3.1 is strictly finer than the compact-open topology, in general.

Moreover, setting functions $f_t, f_0 \in \text{Lip}([0, 1], \mathbb{R})$, where $1/2 \geq t > 0$, as

$$f_t(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq t \\ (x - t)/\sqrt{t} & \text{if } t \leq x \leq 2t \\ \sqrt{t} & \text{if } 2t \leq x \leq 1 \end{cases}$$

and $f_0(x) = 0$ everywhere, the set $K = \{f_t \mid 0 \leq t \leq 1/2\}$ is compact in the compact-open topology, however it is not compact in any topology satisfying (0) and (1) of Theorem 3.1.

Remark 3.3. In [L], Lang considered another topology on the space of all compactly supported Lipschitz real-valued functions on a (locally compact) metric space. We denote it by $\mathfrak{T}_{\text{Lang}}$. His construction also go through for Banach space target. As stated there, one can prove that $\mathfrak{T}_{\text{Lang}}$ satisfies the following property. Let f_j and f be compactly supported Lipschitz functions with $j \in \mathbb{N}$.

$$(3.1) \quad f_j \rightarrow f \text{ in } \mathfrak{T}_{\text{Lang}} \iff \sup_j \text{Lip}(f_j) < \infty \text{ and } f_j \rightarrow f \text{ uniformly.}$$

In particular, if a domain is compact, then $\mathfrak{T}_{\text{Lang}}$ is coarser than our topology. However, the author does not know whether $\mathfrak{T}_{\text{Lang}}$ is metrizable or not.

A topology satisfying Theorem 3.1 which will be given in §5 has a property stronger than (1) of Theorem 3.1. It actually holds that the Lipschitz constant function $\text{Lip}(\cdot)$ on $\text{Lip}_b(Z, X)$ is continuous (Proposition 5.10). Hence, even if a sequence f_j converges to f in $\text{Lip}_b(Z, X)$ uniformly and $\sup_j \text{Lip}(f_j) < \infty$, f_j may diverge in our topology, in general. Indeed, there are functions $f_j : [0, 1] \rightarrow \mathbb{R}$ with $\text{Lip}(f_j) = 1$ for $j \in \mathbb{N}$, such that f_j converges to a constant function uniformly.

Note that if there exists a topology on the space of all (bounded) Lipschitz maps between metric spaces such that it is metrizable and satisfies the above property (3.1), then such a topology obviously satisfies all properties of the conclusion of Theorem 3.1.

The following statements are corollaries to Theorem 3.1.

Corollary 3.4. *If $\phi : X \rightarrow Y$ and $\psi : Z \rightarrow W$ are bi-Lipschitz homeomorphisms, then the map $\text{Lip}_b(W, X) \ni f \mapsto \phi \circ f \circ \psi \in \text{Lip}_b(Z, Y)$ is homeomorphic.*

Corollary 3.5. *Let U be an open set in a metric space X . Let Z be a compact metric space. Then, the topological embedding $\text{Lip}(Z, U) \rightarrow \text{Lip}(Z, X)$ is also an open map.*

Proof. Note that $\{f \in C(Z, X) \mid \text{im } f \subset U\}$ is open in $C(Z, X)$ with respect to the compact-open topology. Since the topology on $\text{Lip}(Z, X)$ is finer than the compact-open topology due to (0) of Theorem 3.1, a set $\{f \in \text{Lip}(Z, X) \mid \text{im } f \subset U\}$ is open in $\text{Lip}(Z, X)$, which is the image of the map $\text{Lip}(Z, U) \rightarrow \text{Lip}(Z, X)$. \square

Corollary 3.6. *Let Z be a compact metric space and X an arbitrary metric space. If a subset \mathcal{K} of $\text{Lip}(Z, X)$ is compact, then the image set $\text{im } \mathcal{K} = \bigcup_{f \in \mathcal{K}} \text{im } f$ is compact.*

Proof. Recall that the evaluation map

$$e : Z \times C(Z, X) \ni (z, f) \mapsto f(z) \in X$$

is continuous in the compact-open topology. Let $\mathcal{K} \subset \text{Lip}(Z, X)$ be a compact set. It is also compact in $C(Z, X)$. Hence, the set $\text{im } \mathcal{K} = e(Z \times \mathcal{K})$ is compact. \square

Corollary 3.7. *Let Z , X and Y be metric spaces. Then, the canonical map*

$$\text{Lip}_b(Z, X) \times \text{Lip}_b(Z, Y) \rightarrow \text{Lip}_b(Z, X \times Y)$$

is homeomorphic.

Proof. This follows from the properties (2), (3) and (4) of Theorem 3.1. \square

Proposition 3.8. *Let Z be a compact metric space and $f : X \rightarrow Y$ a locally Lipschitz map between metric spaces. Then, the map $f_{\#} : \text{Lip}(Z, X) \rightarrow \text{Lip}(Z, Y)$ defined by $g \mapsto f \circ g$ is continuous on each compact set.*

Proof. Let $\mathcal{K} \subset \text{Lip}(Z, X)$ be a compact set. Let us set $X_0 := \bigcup_{g \in \mathcal{K}} \text{im } g$. It is compact due to Corollary 3.6. By Theorem 3.1 (3), the inclusion $\iota : X_0 \hookrightarrow X$ induces a topological embedding $\iota_{\#} : \text{Lip}(Z, X_0) \rightarrow \text{Lip}(Z, X)$. For each $g \in \mathcal{K}$, we define a map $g_0 : Z \rightarrow X_0$ by $g_0(z) = g(z)$ for $z \in Z$. Let us set $\mathcal{K}_0 = \{g_0 \in \text{Lip}(Z, X_0) \mid g \in \mathcal{K}\}$. Then, $\iota_{\#} : \mathcal{K}_0 \rightarrow \mathcal{K}$ is bijective. Since $\iota_{\#}$ is a topological embedding, \mathcal{K}_0 is compact. Thus, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{K}_0 & \hookrightarrow & \text{Lip}(Z, X_0) \\ \downarrow \iota_{\#} & & \downarrow \iota_{\#} \quad \searrow (f|_{X_0})_{\#} \\ \mathcal{K} & \hookrightarrow & \text{Lip}(Z, X) \xrightarrow{f_{\#}} \text{Lip}(Z, Y). \end{array}$$

Since a locally Lipschitz map restricted to a compact set is Lipschitz, $(f|_{X_0})_{\#}$ is continuous by (2) of Theorem 3.1. Let us take an open set O in $\text{Lip}(Z, Y)$. Then, the set

$$\iota_{\#}^{-1}((f_{\#}|_{\mathcal{K}})^{-1}(O)) = \mathcal{K}_0 \cap (f|_{X_0})_{\#}^{-1}(O)$$

is open in \mathcal{K}_0 . Since $\iota_{\#}|_{\mathcal{K}_0}$ is a homeomorphism, the map

$$f_{\#}|_{\mathcal{K}} : \mathcal{K} \rightarrow \text{Lip}(Z, Y)$$

is continuous. □

When a target is a normed abelian group, we obtain

Proposition 3.9. *Let V be a normed abelian group and Z a metric space. Then, $\text{Lip}_b(Z, V)$ is a topological abelian group.*

Proof. Since the addition $+$: $V \times V \rightarrow V$ is Lipschitz, the induced map

$$+_{\#} : \text{Lip}_b(Z, V \times V) \rightarrow \text{Lip}_b(Z, V)$$

is continuous by Theorem 3.1 (2). By Corollary 3.7, the addition operator

$$+ : \text{Lip}_b(Z, V) \times \text{Lip}_b(Z, V) \rightarrow \text{Lip}_b(Z, V)$$

is continuous.

Since $V \ni v \mapsto -v \in V$ is Lipschitz, the map

$$\text{Lip}_b(Z, V) \ni f \mapsto -f \in \text{Lip}_b(Z, V)$$

is continuous. This completes the proof. □

Remark 3.10. Further, a topology satisfying Theorem 3.1 that will be given in §5 satisfies additional properties (Propositions 5.16 and 5.18). However, the properties of Theorem 3.1 is enough to prove Theorem 1.8.

3.4. Lipschitz measure homology. By a similar way to define the measure chain complex, the *Lipschitz measure chain complex* $\mathcal{C}_*^L(X)$ of a metric space X is defined as follows. Let

$$\mathcal{C}_k^L(X) := \mathcal{M}_c(\text{Lip}(\Delta^k, X))$$

for $k \geq 0$. The restriction $r_i : \text{Lip}(\Delta^k, X) \rightarrow \text{Lip}(\Delta^{k-1}, X)$ to the i -th face is continuous for all $i = 0, \dots, k$, due to (3) in Theorem 3.1. Hence, the boundary $\partial : \mathcal{C}_k^L(X) \rightarrow \mathcal{C}_{k-1}^L(X)$ is defined by the same formula as the usual $\partial : \mathcal{C}_k(X) \rightarrow \mathcal{C}_{k-1}(X)$. The *Lipschitz measure homology* of X is defined by

$$\mathcal{H}_*^L(X) := H_*(\mathcal{C}_*^L(X)).$$

Since the inclusion $\text{Lip}(\Delta^k, X) \hookrightarrow C(\Delta^k, X)$ is continuous due to (0) of Theorem 3.1, any measure $\mu \in \mathcal{M}_c(\text{Lip}(\Delta^k, X))$ can be regarded as a measure in $\mathcal{M}_c(C(\Delta^k, X))$ by push-forward. This induces a chain map

$$(3.2) \quad \mathcal{C}_*^L(X) \rightarrow \mathcal{C}_*(X)$$

which is no other than (1.2) in §1.1.

Let Y be another metric space. For a Lipschitz map $\phi : X \rightarrow Y$, the composition $\phi_\# : \text{Lip}(\Delta^k, X) \rightarrow \text{Lip}(\Delta^k, Y)$ is continuous, due to (2) of Theorem 3.1. Then, we can define the push-forward

$$\phi_{\#\#} : \mathcal{C}_k^L(X) \rightarrow \mathcal{C}_k^L(Y)$$

of measures by the continuous map $\phi_\#$. The map $\phi_{\#\#}$ will be written by $\phi_\#$ for shortly. It is actually a chain map, due to the definition of the boundary.

Further, for a *locally* Lipschitz map $\phi : X \rightarrow Y$, we can define the push-forward $\phi_\# : \mathcal{C}_*^L(X) \rightarrow \mathcal{C}_*^L(Y)$ as follows. Let $\mathcal{K} \subset \text{Lip}(\Delta^k, X)$ be a compact determination set of $\mu \in \mathcal{C}_k^L(X)$. The restriction of the composition $\phi_\# : \text{Lip}(\Delta^k, X) \rightarrow \text{Lip}(\Delta^k, Y)$; $\sigma \mapsto \phi \circ \sigma$ to \mathcal{K} is continuous, due to Proposition 3.8. Since the measure μ is essentially defined on \mathcal{K} , the push-forward of $\mu|_{\mathcal{K}}$ under the continuous map $\phi_\#|_{\mathcal{K}}$ is defined. We denote it by $\phi_\#\mu$. This construction is actually well-defined due to the following

Lemma 3.11. *Let $\phi : X \rightarrow Y$ be a locally Lipschitz map between metric spaces. Then, the above construction of the push-forward $\mathcal{C}_k^L(X) \ni \mu \mapsto \phi_\#\mu \in \mathcal{C}_k^L(Y)$ does not depend on the choice of a compact determination set \mathcal{K} of μ . Further, $\phi_\# : \mathcal{C}_*^L(X) \rightarrow \mathcal{C}_*^L(Y)$ is a chain map.*

Proof. Let \mathcal{K}_1 and \mathcal{K}_2 be two compact determination sets of μ in $\text{Lip}(\Delta^k, X)$. Since a set containing a determination set is a determination set, $\mathcal{K}_1 \cup \mathcal{K}_2$ is also a compact determination set. Hence, we may assume that $\mathcal{K}_1 \subset \mathcal{K}_2$. Let us consider the push-forwards $\nu_i := (\phi_\#|_{\mathcal{K}_i})_\#(\mu|_{\mathcal{K}_i})$ of $\mu|_{\mathcal{K}_i} \in \mathcal{M}_c(\mathcal{K}_i)$ by the continuous maps

$\phi_{\#}|_{\mathcal{K}_i} : \mathcal{K}_i \rightarrow \text{Lip}(\Delta^k, Y)$ for $i = 1, 2$. Then, $\nu_i \in \mathcal{M}_c(\phi_{\#}(\mathcal{K}_i))$. Further, we regard them as signed Borel measures on $\text{Lip}(\Delta^k, Y)$. Let \mathcal{A} be a Borel set in $\text{Lip}(\Delta^k, Y)$. Then, we obtain

$$\nu_2(\mathcal{A}) = \mu((\phi_{\#}|_{\mathcal{K}_2})^{-1}(\mathcal{A})) = \mu(\phi_{\#}^{-1}(\mathcal{A}) \cap \mathcal{K}_2).$$

Since \mathcal{K}_1 is a determination set of μ , we have

$$\begin{aligned} \mu(\phi_{\#}^{-1}(\mathcal{A}) \cap \mathcal{K}_2) &= \mu(\phi_{\#}^{-1}(\mathcal{A}) \cap \mathcal{K}_2 \cap \mathcal{K}_1) + \mu(\phi_{\#}^{-1}(\mathcal{A}) \cap \mathcal{K}_2 - \mathcal{K}_1) \\ &= \mu(\phi_{\#}^{-1}(\mathcal{A}) \cap \mathcal{K}_1) \\ &= \nu_1(\mathcal{A}). \end{aligned}$$

Therefore, $\nu_1 = \nu_2$ as the signed Borel measures on $\text{Lip}(\Delta^k, Y)$, which are denoted by $\phi_{\#}\mu$. By the construction, $\phi_{\#}\mu$ is of finite total variation and has a compact determination set.

It is easily check that the map $\phi_{\#} : \mathcal{C}_*^L(X) \rightarrow \mathcal{C}_*^L(Y)$ is a chain map. This completes the proof. \square

We can easily see that the map (3.2) is natural, i.e., $\phi_{\#} \circ (\mathcal{C}_*^L(X) \rightarrow \mathcal{C}_*(X)) = (\mathcal{C}_*^L(Y) \rightarrow \mathcal{C}_*(Y)) \circ \phi_{\#}$ holds, for every locally Lipschitz map $\phi : X \rightarrow Y$. Therefore, it induces a natural transformation

$$\mathcal{H}_*^L \rightarrow \mathcal{H}_*$$

on the category of all metric spaces and all locally Lipschitz maps.

For a pair (X, A) of metric spaces, the inclusion $A \hookrightarrow X$ induces a topological embedding $\text{Lip}(\Delta^k, A) \rightarrow \text{Lip}(\Delta^k, X)$ for every $k \geq 0$, due to (2) of Theorem 3.1. So, it also induces an injective chain map

$$\mathcal{C}_*^L(A) \rightarrow \mathcal{C}_*^L(X).$$

Thus, $\mathcal{C}_*^L(A)$ is regarded as a subcomplex of $\mathcal{C}_*^L(X)$. We set

$$\mathcal{C}_*^L(X, A) = \mathcal{C}_*^L(X) / \mathcal{C}_*^L(A)$$

and call it the Lipschitz measure complex of (X, A) . Its homology is denoted by

$$\mathcal{H}_*^L(X, A)$$

called the Lipschitz measure homology of (X, A) . There are also natural maps

$$\mathcal{C}_*^L(X, A) \rightarrow \mathcal{C}_*(X, A)$$

and

$$\mathcal{H}_*^L(X, A) \rightarrow \mathcal{H}_*(X, A)$$

on the category of all pairs of metric spaces and all locally Lipschitz maps.

Remark 3.12. By (5) of Theorem 3.1, there are canonical identifications $C(\Delta^0, X) = \text{Lip}(\Delta^0, X) = X$ for any metric space X . Hence, we identify $\mathcal{C}_0(X)$ and $\mathcal{C}_0^L(X)$ with $\mathcal{M}_c(X)$.

The following is independent on main results.

Proposition 3.13. *Let X be a metric space which has no nonconstant Lipschitz curves. Namely, if $\sigma : [0, 1] \rightarrow X$ is Lipschitz, then σ is a constant map. Then, $\mathcal{H}_k^L(X) = 0$ for all $k \geq 1$.*

Proof. Let X be assumed as in the assumption. Then, $\text{Lip}(\Delta^k, X) = \text{Lip}(\Delta^0, X) = X$ for all $k \geq 1$. Hence, $\mathcal{C}_k^L(X) = \mathcal{C}_0^L(X) = \mathcal{M}_c(X)$ for $k \geq 1$. The boundary map $\partial = (\partial_k)_{k \geq 1}$ of the complex $\mathcal{C}_*^L(X)$ becomes

$$\partial_k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \text{id}_{\mathcal{M}_c(X)} & \text{if } k \text{ is even.} \end{cases}$$

This implies $\mathcal{H}_k^L(X) = 0$ for any $k \geq 1$. \square

3.5. A natural map from \mathcal{C}_*^L to \mathbf{N}_*^c . In this subsection, we construct a natural chain map from the Lipschitz measure chain complex to the current chain complex.

Let us denote by X a metric space. For $\sigma \in \text{Lip}(\Delta^k, X)$, Riedweg and Schäppi ([RS]) considered a functional $[\sigma] : \mathcal{D}^k(X) \rightarrow \mathbb{R}$ defined by

$$[\sigma](fd\pi) = \int_{\Delta^k} f \circ \sigma(s) \det(\nabla(\pi \circ \sigma(s))) d\mathcal{L}^k(s)$$

for each $fd\pi \in \mathcal{D}^k(X)$, where \mathcal{L}^k is the k -dimensional Lebesgue measure. Here, the gradient $\nabla(\pi \circ \sigma)$ is defined for almost all points on Δ^k , due to Rademacher's theorem. It is also represented as $[\sigma] = \sigma_{\#} \llbracket 1_{\Delta^k} \rrbracket$, where $\llbracket 1_{\Delta^k} \rrbracket$ is a k -current in Δ^k given by

$$\llbracket 1_{\Delta^k} \rrbracket(gd\tau) = \int_{\Delta^k} g \det(\nabla\tau) d\mathcal{L}^k$$

for $gd\tau \in \mathcal{D}^k(\Delta^k)$ (see [AK]). By the definition, $[\sigma]$ is a normal k -current having compact support contained in the image of σ .

Lemma 3.14. *For each $fd\pi \in \mathcal{D}^k(X)$, the functional*

$$\text{Lip}(\Delta^k, X) \ni \sigma \mapsto [\sigma](fd\pi) \in \mathbb{R}$$

is continuous in the topology on $\text{Lip}(\Delta^k, X)$.

Proof. Let us fix $fd\pi \in \mathcal{D}^k(X)$. By Theorem 3.1 (0), $\text{Lip}(\Delta^k, X)$ is metrizable. Hence, it suffices to show that $[\cdot](fd\pi)$ is sequentially continuous. Let a sequence σ_j converge to σ in $\text{Lip}(\Delta^k, X)$. By Theorem 3.1 (1), $f \circ \sigma_j \rightarrow f \circ \sigma$ and $\pi \circ \sigma_j \rightarrow \pi \circ \sigma$ uniformly as $j \rightarrow \infty$, and $\sup_j \text{Lip}(\pi \circ \sigma_j) < \infty$. Hence, we have

$$[\sigma_j](fd\pi) = \llbracket 1_{\Delta^k} \rrbracket(f \circ \sigma_j d\pi \circ \sigma_j) \rightarrow \llbracket 1_{\Delta^k} \rrbracket(f \circ \sigma d\pi \circ \sigma) = [\sigma](fd\pi),$$

as $j \rightarrow \infty$, because $\llbracket 1_{\Delta^k} \rrbracket$ is a current. \square

Remark 3.15. The functional $[\cdot](fd\pi)$ is not continuous in general, when $\text{Lip}(\Delta^k, X)$ is merely equipped with the compact-open topology.

Let us consider a family $\{u_\epsilon\}_{\epsilon > 0}$ of smooth maps from Δ^k to \mathbb{R}^k , for $k \geq 2$ such that $u_\epsilon \rightarrow u$ as $\epsilon \rightarrow 0$ uniformly, and $\det(\nabla u_\epsilon) \rightarrow 1$ in

$L^\infty(\Delta^k)$ weakly*, as $\epsilon \rightarrow 0$, where $u(x) = 0 \in \mathbb{R}^k$ for $x \in \Delta^k$. Here, Δ^k is considered as a convex subset of \mathbb{R}^k . Such a family can be found in [ABF], actually is given by

$$\begin{cases} (u_\epsilon)_1(x) = \sqrt{2\epsilon} \sin(x_1/\epsilon), \\ (u_\epsilon)_2(x) = x_2 \sqrt{2\epsilon} \cos(x_1/\epsilon), \\ (u_\epsilon)_i(x) = x_i \quad (3 \leq i \leq k). \end{cases}$$

Obviously, $\det(\nabla u) = 0$. Note that each u_ϵ is Lipschitz and

$$\sup_{\epsilon > 0} \text{Lip}(u_\epsilon) = \infty.$$

By using this family, we show the discontinuity of $[\cdot](fd\pi)$ on $\text{Lip}(\Delta^k, \mathbb{R}^k)$ in the compact-open topology, for a particular choice of $fd\pi \in \mathcal{D}^k(\mathbb{R}^k)$. Indeed, we choose $f(x) = 1$ and $\pi(x) = x$ for $x \in \mathbb{R}^k$. Then, we have

$$[u_\epsilon](fd\pi) = \int_{\Delta^k} \det(\nabla u_\epsilon) d\mathcal{L}^k \rightarrow \mathcal{L}^k(\Delta^k) \quad \text{as } \epsilon \rightarrow 0,$$

and $[u](fd\pi) = 0$. Hence, $\lim_{\epsilon \rightarrow 0} [u_\epsilon](fd\pi) \neq [u](fd\pi)$ actually holds.

Definition 3.16. For $\mu \in \mathcal{C}_k^L(X)$, we define a functional $T^\mu : \mathcal{D}^k(X) \rightarrow \mathbb{R}$ by

$$(3.3) \quad T^\mu(fd\pi) = \int_{\text{Lip}(\Delta^k, X)} [\sigma](fd\pi) d\mu$$

for $fd\pi \in \mathcal{D}^k(X)$. Due to Lemma 3.14, the integral (3.3) is well-defined.

Remark 3.17. Let us denote by \mathfrak{T}_{CO} the compact-open topology on $\text{Lip}(\Delta^k, X)$. We observe that the integral (3.3) can not be defined for a finite Borel measure on $(\text{Lip}(\Delta^k, X), \mathfrak{T}_{\text{CO}})$ with compact support.

We consider smooth functions $v_\epsilon, v_0 \in C^1(\Delta^2, \mathbb{R}^2)$ for $\epsilon > 0$, similar to functions appeared in Remark 3.15, defined by

$$\begin{cases} (v_\epsilon)_1(x) = \sqrt{\epsilon} \sin(x_1/\epsilon^2) \\ (v_\epsilon)_2(x) = \sqrt{\epsilon} x_2 \cos(x_1/\epsilon^2), \end{cases}$$

and $v_0 = 0$. Here, we identify Δ^2 with $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in [-1, 1], x_2 \in [0, \sqrt{3}x_1]\}$. Then, the map

$$v_\bullet : [0, 1] \ni \epsilon \mapsto v_\epsilon \in C^1(\Delta^2, \mathbb{R}^2) \subset \text{Lip}(\Delta^2, \mathbb{R}^2)$$

is continuous in \mathfrak{T}_{CO} . Hence, its image $K := \{v_\epsilon \mid 0 \leq \epsilon \leq 1\}$ is compact in \mathfrak{T}_{CO} . We consider the push-forward of the Lebesgue measure \mathcal{L}^1 on $[0, 1]$ under the map v_\bullet , and set it as

$$\mu := (v_\bullet)_\# \mathcal{L}^1.$$

By the definition, μ is a finite positive Borel measure on $(\text{Lip}(\Delta^2, \mathbb{R}^2), \mathfrak{T}_{\text{CO}})$ supported in the compact set K . For $fd\pi \in \mathcal{D}^2(\mathbb{R}^2)$ with $f = 1$ and $\pi = \text{id}$, we check that $[\cdot](fd\pi)$ can not be integrable in μ . Suppose that

$$\text{Lip}(\Delta^2, \mathbb{R}^2) \ni \sigma \mapsto [\sigma](fd\pi) \in \mathbb{R}$$

is Borel measurable in \mathfrak{T}_{CO} . Then, we have

$$\begin{aligned} \int_{\text{Lip}(\Delta^2, \mathbb{R}^2)} [\sigma](f d\pi) d\mu(\sigma) &= \int_0^1 [v_\epsilon](f d\pi) d\epsilon \\ &= \int_0^1 \int_{\Delta^2} \det(\nabla v_\epsilon)(x) dx d\epsilon \\ &= C \int_0^1 \frac{d\epsilon}{\epsilon} + C', \end{aligned}$$

where C and C' are constants with $C > 0$. Therefore, the integral of $[\cdot](f d\pi)$ is not integrable in μ .

Remark 3.18. In [PS1] and [PS2], Paolini and Stepanov thoroughly researched one-dimensional normal currents in arbitrary metric spaces. For a metric space E , they provided a pseudo-distance function d_Θ on $\text{Lip}([0, 1], E)$ which is related to the uniform distance. The quotient space of $\text{Lip}([0, 1], E)$ under the relation $d_\Theta = 0$ was denoted by $\Theta(E)$. By the definition of d_Θ , it is known that if $d_\Theta(\theta, \theta') = 0$, then $[\theta] = [\theta']$ holds. Their results said that *every* normal one-dimensional current in *every* metric space E is represented by the integral of $[\theta]$ in a positive Borel measure on $\Theta(E)$ (and in a positive Borel measure on $C([0, 1], E)$ concentrated on $\text{Lip}([0, 1], E)$), vice versa. See [PS1] and [PS2] for more details. In contrast to their results, our Theorem 1.8 says that every normal current *cycle* (i.e., its boundary is zero) of *every dimension*, in every WLLC metric space is represented by an integral of $[\theta]$.

Theorem 3.19. *Let X be an arbitrary metric space. For $\mu \in \mathcal{C}_k^{\text{L}}(X)$, T^μ is a normal k -current in X of compact support. Further, $\mathcal{C}_*^{\text{L}}(X) \ni \mu \mapsto T^\mu \in \mathbf{N}_*^c(X)$ is a chain map.*

Proof. Let us take $\mu \in \mathcal{C}_k^{\text{L}}(X)$. Let \mathcal{K} denote its compact determination set in $\text{Lip}(\Delta^k, X)$. By Corollary 3.6, the image set $\text{im } \mathcal{K}$ is compact. We first prove that T^μ is a k -current in X . By the definition, T^μ is multilinear on $\mathcal{D}^k(X)$. Since each $[\sigma]$ satisfies the locality, T^μ satisfies the locality. To check that T^μ has finite mass, we estimate the absolute value $|T^\mu(f d\pi)|$ as follows.

$$\begin{aligned} |T^\mu(f d\pi)| &\leq \int_{\mathcal{K}} \int_{\Delta^k} |f \circ \sigma \det(\nabla(\pi \circ \sigma))| d\mathcal{L}^k d|\mu| \\ &\leq \text{Lip}(\sigma)^k \prod_{i=1}^k \text{Lip}(\pi_i) \int_{\mathcal{K}} \int_{\Delta^k} |f \circ \sigma| d\mathcal{L}^k d|\mu| \\ &\leq L^k \prod_{i=1}^k \text{Lip}(\pi_i) \int_{\mathcal{K}} \int_{\Delta^k} |f \circ \sigma| d\mathcal{L}^k d|\mu|. \end{aligned}$$

Here, $L := \sup_{\sigma \in \mathcal{K}} \text{Lip}(\sigma)$, which is finite by (1) of Theorem 3.1. Now, we note that the map

$$\mathcal{K} \ni \sigma \mapsto \int_{\Delta^k} g \circ \sigma d\mathcal{L}^k$$

is continuous for each $g \in C(\text{im } \mathcal{K})$, where $C(\text{im } \mathcal{K})$ is the space of all continuous functions from $\text{im } \mathcal{K}$ to \mathbb{R} . From this, we can consider a functional

$$C(\text{im } \mathcal{K}) \ni g \mapsto \int_{\mathcal{K}} \int_{\Delta^k} g \circ \sigma d\mathcal{L}^k d|\mu| \in \mathbb{R}.$$

Since this functional is positive linear, it is represented by the integral over some positive Borel measure ν on $\text{im } \mathcal{K}$, due to the Riesz-Markov-Kakutani representation theorem. We regard the measure ν as a measure on X supported in the compact set $\text{im } \mathcal{K}$. Then, we obtain

$$|T^\mu(f d\pi)| \leq L^k \prod_{i=1}^k \text{Lip}(\pi_i) \int_X |f| d\nu$$

for every $f d\pi \in \mathcal{D}^k(X)$. Therefore, T^μ is proved to have finite mass. Next, we prove that T^μ is continuous in the sense of current. Let us take sequences $\pi_i^j \in \text{Lip}(X)$ converging to π_i as $j \rightarrow \infty$ for $i = 1, \dots, k$ with $\sup_{i,j} \text{Lip}(\pi_i^j) < \infty$. We set $L' := \sup_{i,j} \text{Lip}(\pi_i^j)$. Then, $[\cdot](f d\pi^j)$ converges to $[\cdot](f d\pi)$ pointwise on $\text{Lip}(\Delta^k, X)$ as $j \rightarrow \infty$. Further, we have

$$\begin{aligned} \left| \int_{\mathcal{K}} [\sigma](f d\pi^j) d|\mu| \right| &\leq \int_{\mathcal{K}} |[\sigma](f d\pi^j)| d|\mu| \\ &\leq \int_{\mathcal{K}} \int_{\Delta^k} |f \circ \sigma \det(\nabla(\pi^j \circ \sigma))| d\mathcal{L}^k d|\mu| \\ &\leq (LL')^k \int_X |f| d\nu \end{aligned}$$

for all $j \in \mathbb{N}$. Therefore, by the dominated convergence theorem,

$$\lim_{j \rightarrow \infty} T^\mu(f d\pi^j) = T^\mu(f d\pi)$$

holds. Thus, the continuity of T^μ is proved.

To prove that T^μ is normal, let us compare ∂T^μ with $T^{\partial\mu}$. For any $f d\pi \in \mathcal{D}^{k-1}(X)$, we have

$$\begin{aligned} \partial T^\mu(f d\pi) &= T^\mu(df \wedge d\pi) = \int_{\text{Lip}(\Delta^k, X)} [\sigma](df \wedge d\pi) d\mu \\ &= \int_{\text{Lip}(\Delta^k, X)} [\partial\sigma](f d\pi) d\mu = \sum_{i=0}^k (-1)^i \int_{\text{Lip}(\Delta^k, X)} [r_i \sigma](f d\pi) d\mu, \end{aligned}$$

where r_i denotes the restriction to the i -th face. On the other hands, we obtain

$$\begin{aligned} T^{\partial\mu}(fd\pi) &= \int_{\text{Lip}(\Delta^{k-1}, X)} [\tau](fd\pi) d(\partial\mu) \\ &= \sum_{i=0}^k (-1)^i \int_{\text{Lip}(\Delta^{k-1}, X)} [\tau](fd\pi) d(r_{i\#}\mu) \\ &= \sum_{i=0}^k (-1)^i \int_{\text{Lip}(\Delta^k, X)} [r_i\sigma](fd\pi) d\mu. \end{aligned}$$

Therefore, $\partial T^\mu = T^{\partial\mu}$, and hence, T^μ is normal. \square

The following is fundamental and important.

Lemma 3.20. *Let X be an arbitrary metric space. The map $\mathcal{C}_0^L(X) \ni \mu \mapsto T^\mu \in \mathbf{N}_0^c(X)$ is isomorphic.*

Proof. We prove that the map $T^\bullet : \mu \mapsto T^\mu$ is surjective. Let us take $T \in \mathbf{N}_0^c(X)$. Let K denote its compact support. Then, $T|_K$ and T are same as currents. Here, $T|_K$ is the linear functional defined by

$$T|_K(f) = T(1_K f)$$

for all bounded Borel functions $f : X \rightarrow \mathbb{R}$, where 1_K is the characteristic function of K . Further, $T|_K$ is also considered as a normal current in K . Due to the finite mass axiom, $T = T|_K$ is continuous on the space $C(K)$ with respect to the uniform topology. By the Riesz-Markov-Kakutani theorem, there is a unique regular Borel measure μ on K such that

$$T|_K(f) = \int_K f d\mu$$

holds for all $f \in C(K)$, and that the total variation of μ equals the operator norm of T as the linear functional $T : C(K) \rightarrow \mathbb{R}$.

Let us regard μ as a signed Borel measure $\bar{\mu}$ on X . Namely, it is defined by

$$\bar{\mu}(A) = \mu(A \cap K)$$

for every Borel set A of X . Obviously, $\bar{\mu}$ has the compact determination set K and is of finite total variation. Namely, $\bar{\mu} \in \mathcal{M}_c(X) = \mathcal{C}_0(X) = \mathcal{C}_0^L(X)$. Then, for every $f \in \text{Lip}_b(X)$, we have

$$T^{\bar{\mu}}(f) = \int_X f d\bar{\mu} = \int_K f d\mu = T|_K(f|_K) = T(f).$$

Therefore, T^\bullet is surjective.

If $T^\mu = 0$, then obviously $\mu = 0$. Hence, the map T^\bullet is injective. \square

For the proof of Theorem 1.8, we need only the surjectivity of $\mathcal{C}_0^L \rightarrow \mathbf{N}_0^c$.

4. PROOF OF THEOREM 1.8

In this section, we prove Theorem 1.8 by using cosheaf theory following [B]. First, we recall the definition of cosheaves and its fundamental properties. We prove that the functor taking measure (resp. Lipschitz measure) chain complex on each open set of a topological (resp. metric) space is a cosheaf. Theorem 4.1 gives a general way to show a coincidence of two homologies associated to cosheaves on a space. Finally, checking that these cosheaves satisfy the assumption of Theorem 4.1, we obtain the proof of Theorem 1.8.

4.1. Review of cosheaf theory. Let us recall the notion of cosheaf. Let X be a topological space and $\mathcal{O}(X)$ the set of all open sets in X . We regard $\mathcal{O}(X)$ as a category, by assigning an arrow $U \rightarrow V$ if and only if $U \subset V$. Let \mathbf{Ch} denote the category of chain complexes of real vector spaces and chain maps. Here, we consider chain complexes indexed \mathbb{Z} . A covariant functor $\mathfrak{A} : \mathcal{O}(X) \rightarrow \mathbf{Ch}$ is called a *precosheaf* (valued in \mathbf{Ch}). We denote the map $\mathfrak{A}(U \rightarrow V) : \mathfrak{A}(U) \rightarrow \mathfrak{A}(V)$ by $i_{V,U}$. The precosheaf \mathfrak{A} is called a *cosheaf* if, for any family $\{U_\alpha\}$ of open sets in X ,

$$\bigoplus_{\alpha, \beta} \mathfrak{A}(U_\alpha \cap U_\beta) \xrightarrow{\Phi_1} \bigoplus_{\alpha} \mathfrak{A}(U_\alpha) \xrightarrow{\Phi_0} \mathfrak{A}(U) \rightarrow 0$$

is exact, where $U = \bigcup_{\alpha} U_\alpha$, $\Phi_0 = \sum_{\alpha} i_{U, U_\alpha}$ and $\Phi_1 = \sum_{\alpha, \beta} i_{U_\beta, U_\alpha \cap U_\beta} - i_{U_\alpha, U_\alpha \cap U_\beta}$.

Let \mathbf{Vect} denote the category of all vector spaces and all linear maps over \mathbb{R} . We define two homology functors $H_m, \tilde{H}_m : \mathbf{Ch} \rightarrow \mathbf{Vect}$ for each $m \geq 0$ as follows. Let $C = (C_k)_{k \in \mathbb{Z}}$ be a chain complex. Then, $H_m(C)$ denotes the m -th homology of the restricted chain complex $(C_k)_{k \geq 0}$ and $\tilde{H}_m(C)$ denotes the usual m -th homology of C :

$$H_m(C) := \begin{cases} \ker \partial / \operatorname{im} \partial & \text{if } m \geq 1 \\ C_0 / \operatorname{im} \partial & \text{if } m = 0, \end{cases}$$

$$\tilde{H}_m(C) := \ker \partial / \operatorname{im} \partial \quad \text{for any } m.$$

For a precosheaf $\mathfrak{A} : \mathcal{O}(X) \rightarrow \mathbf{Ch}$, we have precosheaves $H_m(\mathfrak{A}), \tilde{H}_m(\mathfrak{A}) : \mathcal{O}(X) \rightarrow \mathbf{Vect}$ for all $m \geq 0$.

A precosheaf \mathfrak{A}'' on X is said to be *flabby* if $\mathfrak{A}''(O) \rightarrow \mathfrak{A}''(X)$ is injective for all $O \in \mathcal{O}(X)$. The precosheaf \mathfrak{A}'' is said to be *locally trivial* if for any $x \in X$ and $O \in \mathcal{O}(X)$ with $x \in O$, there is $O' \in \mathcal{O}(X)$ with $x \in O' \subset O$ such that the map $i_{O, O'} : \mathfrak{A}''(O') \rightarrow \mathfrak{A}''(O)$ is zero.

We present a sufficient condition when two precosheaves $H_*(\mathfrak{A})$ and $H_*(\mathfrak{A}')$ coincide with, for cosheaves \mathfrak{A} and \mathfrak{A}' valued in \mathbf{Ch} . In [Mi], the author essentially proved the following

Theorem 4.1 ([Mi]). *Let X be a paracompact topological space. Let \mathfrak{A} and \mathfrak{A}' be precosheaves on X valued in \mathbf{Ch} . Suppose the following conditions:*

- (i) the precosheaves \mathfrak{A}_m and \mathfrak{A}'_m are flabby cosheaves valued in \mathbf{Vect} for $m \geq 0$,
- (ii) there exists a natural transformation $\phi : \mathfrak{A} \rightarrow \mathfrak{A}'$ such that $\mathfrak{A}_0(O) \rightarrow \mathfrak{A}'_0(O)$ is surjective and $\mathfrak{A}_{-1}(O) \rightarrow \mathfrak{A}'_{-1}(O)$ is injective for each $O \in \mathcal{O}(X)$,
- (iii) the precosheaves $\tilde{H}_m(\mathfrak{A})$ and $H_n(\mathfrak{A}')$ are locally trivial on X for all $m \geq 0$ and $n \geq 1$.

Then, ϕ induces an isomorphism

$$\phi_* : H_m(\mathfrak{A}) \rightarrow H_m(\mathfrak{A}')$$

between precosheaves for each $m \geq 0$.

One can find out the proof of Theorem 4.1 in [Mi].

The following statement is convenience.

Proposition 4.2 ([B, Chap. VI]). *Let \mathfrak{A} be a precosheaf on a topological space X . Then, \mathfrak{A} is a cosheaf if and only if it satisfies the following two conditions:*

- For any two open sets U and V in X , the short sequence

$$\mathfrak{A}(U \cap V) \xrightarrow{\Phi_1} \mathfrak{A}(U) \oplus \mathfrak{A}(V) \xrightarrow{\Phi_0} \mathfrak{A}(U \cup V) \rightarrow 0$$

is exact, where $\Phi_1 = -i_{U, U \cap V} + i_{V, U \cap V}$ and $\Phi_0 = i_{U \cup V, U} + i_{U \cup V, V}$.

- Let $\{U_\alpha\}$ be a family of open sets of X , directed upwards by inclusion. Namely, for any indices α, α' , there exists α'' such that $U_\alpha \cup U_{\alpha'} \subset U_{\alpha''}$. We set $U = \bigcup_\alpha U_\alpha$. Then, the maps $i_{U, U_\alpha} : \mathfrak{A}(U_\alpha) \rightarrow \mathfrak{A}(U)$ induces an isomorphism

$$\varinjlim i_{U, U_\alpha} : \varinjlim \mathfrak{A}(U_\alpha) \rightarrow \mathfrak{A}(U).$$

4.2. \mathcal{C}_* and \mathcal{C}_*^L as cosheaves. In this section, we prove that the functors \mathcal{C}_* and \mathcal{C}_*^L are flabby cosheaves on a metric space valued in chain complexes.

Lemma 4.3. *Let S be a metrizable space. Then, the following holds.*

- (1) For a compact set K and an open set U in S with $K \subset U$, there is an open set U' in S such that $K \subset U' \subset \overline{U'} \subset U$. Here, $\overline{U'}$ is the closure of U' in the topology of S .
- (2) For a compact set K and open sets U and V in S with $K \subset U \cup V$, there exists an open set W such that $\overline{W} \subset V$ and $K - W \subset U$.

Proof. We fix a metric on S which is compatible to the topology. We prove (1). Let U be an open set in S and K a compact set with $K \subset U$. Let us consider the distance function ρ from $X - U$. Then, $\{\rho > 0\} = U$. Since K is compact, $\epsilon := \min_K \rho > 0$. Let us set $U' := \{\rho > \epsilon/2\}$. Then, it satisfies the desired property of (1).

We prove (2). Let U and V be open sets in S and K a compact set with $K \subset U \cup V$. Let us consider the distance function ρ from $S - V$.

Then, there exists $\epsilon > 0$ such that $K \cap \{\rho \leq \epsilon\} \subset U$. Indeed, if this claim is false, there exists a sequence $x_j \in K \cap \{\rho \leq 1/j\}$ and $x_j \notin U$. Some subsequence of x_j converges to some point x . Then, the point x satisfies $x \in K$, $x \notin U$ and $x \notin V$, which is a contradiction. We set $W = \{\rho > \epsilon\}$. Then, it satisfies the desired property of (2). \square

Lemma 4.4. *Let S be a metrizable space. Then, for any $U, V \in \mathcal{O}(S)$,*

$$\mathcal{M}_c(U \cap V) \xrightarrow{\Phi_1} \mathcal{M}_c(U) \oplus \mathcal{M}_c(V) \xrightarrow{\Phi_0} \mathcal{M}_c(U \cup V) \rightarrow 0$$

is exact, where $\Phi_1(\xi) = (\xi, -\xi)$ and $\Phi_0(\mu, \nu) = \mu + \nu$.

Proof. Let $\xi \in \mathcal{M}_c(U \cup V)$ with a compact determination set $K \subset U \cup V$. By Lemma 4.3, there exists an open set W in S such that

$$\overline{W} \subset V \text{ and } K - W \subset U.$$

The restriction $\mu := \xi \lfloor (K - W)$ can be regarded as a signed measure on U with the compact determination set $K - W$. Let us consider the restriction $\nu := \xi \lfloor (K \cap W)$ which is regarded as a signed measure on V . It has a determination set $K \cap W$. Since a set containing a determination set is also a determination set, the compact set $K \cap \overline{W}$ is also a determination set of ν . Therefore, we have $\mu \in \mathcal{M}_c(U)$ and $\nu \in \mathcal{M}_c(V)$ with $\mu + \nu = \xi$. Hence, Φ_0 is surjective.

Let us take $\mu \in \mathcal{M}_c(U)$ and $\nu \in \mathcal{M}_c(V)$ with $\mu + \nu = 0 \in \mathcal{M}_c(U \cup V)$. Let K and L be compact sets in U and V which are determination sets of μ and ν , respectively. By Lemma 4.3, there is an open subset V' in S such that

$$L \subset V' \subset \overline{V'} \subset V.$$

Then, $\mu \lfloor (K - V') = 0$ in $\mathcal{M}_c(U)$. We set $\xi := \mu \lfloor \overline{V'}$ which is a signed measure on the compact set $K \cap \overline{V'}$. Thus, we can regard ξ as an element of $\mathcal{M}_c(U \cap V)$. By the construction, we have $\Phi_1(\xi) = (\mu, \nu)$. \square

Lemma 4.5. *Let $\{U_\alpha\}$ be a family of open sets in a topological space S which is directed upwards by inclusions. Then, the map $\varinjlim \mathcal{M}_c(U_\alpha) \rightarrow \mathcal{M}_c(U)$ induced by the maps $\mathcal{M}_c(U_\alpha) \rightarrow \mathcal{M}_c(U)$, is isomorphic, where $U = \bigcup_\alpha U_\alpha$.*

Proof. Since all maps $\mathcal{M}_c(U_\alpha) \rightarrow \mathcal{M}_c(U)$ are injective, the direct limit map $\varinjlim \mathcal{M}_c(U_\alpha) \rightarrow \mathcal{M}_c(U)$ is injective. Let us take $\mu \in \mathcal{M}_c(U)$ with a compact determination set $K \subset U$. Since $\{U_\alpha\}$ is directed upwards by inclusions, there is α such that $K \subset U_\alpha$. Then, we can regard μ as a measure in $\mathcal{M}_c(U_\alpha)$. Hence, the map $\varinjlim \mathcal{M}_c(U_\alpha) \rightarrow \mathcal{M}_c(U)$ is surjective. \square

Corollary 4.6. *For a topological space X , the correspondence $\mathcal{O}(X) \ni O \mapsto \mathcal{C}_*(O) \in \mathbf{Ch}$ is a flabby cosheaf. For a metric space X , the correspondence $\mathcal{O}(X) \ni O \mapsto \mathcal{C}_*^L(O) \in \mathbf{Ch}$ is a flabby cosheaf.*

Proof. This follows from Theorem 3.1, Corollary 3.5, Lemmas 4.4 and 4.5 and Proposition 4.2. \square

4.3. Reduced homologies. Let us consider an augmentation of the measure chain complex of a metric space X defined by

$$\tilde{\partial}_0 : \mathcal{C}_0(X) = \mathcal{M}_c(X) \ni \mu \mapsto \mu(X) \in \mathbb{R}.$$

For a metric space Y , an augmentation of the Lipschitz measure chain complex is also defined by $\tilde{\partial}_0 : \mathcal{C}_0^L(Y) \ni \mu \mapsto \mu(Y) \in \mathbb{R}$.

Lemma 4.7. *The maps $\tilde{\partial}_0$ are actually augmentations of $\mathcal{C}_*(X)$ and $\mathcal{C}_*^L(Y)$ for a topological space X and a metric space Y .*

Proof. Let $\mu \in \mathcal{C}_1(X)$. Then,

$$\tilde{\partial}_0 \partial_1 \mu = \partial_1 \mu(X) = \int_X 1 d(\partial_1 \mu) = \int_{C(I, X)} 1 - 1 d\mu = 0.$$

Similarly, we have $\tilde{\partial}_0 \partial_1 \mu = 0$ for $\mu \in \mathcal{C}_1^L(Y)$. \square

The augmented (Lipschitz) measure chain complexes by $\tilde{\partial}_0$ are denoted by $\tilde{\mathcal{C}}_*$ and $\tilde{\mathcal{C}}_*^L$. Namely,

$$\begin{aligned} \tilde{\mathcal{C}}_* &= (\cdots \xrightarrow{\partial_{k+1}} \mathcal{C}_k \xrightarrow{\partial_k} \mathcal{C}_{k-1} \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_1} \mathcal{C}_0 \xrightarrow{\tilde{\partial}_0} \mathbb{R}) \\ \tilde{\mathcal{C}}_*^L &= (\cdots \xrightarrow{\partial_{k+1}} \mathcal{C}_k^L \xrightarrow{\partial_k} \mathcal{C}_{k-1}^L \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_1} \mathcal{C}_0^L \xrightarrow{\tilde{\partial}_0} \mathbb{R}). \end{aligned}$$

Their homologies are written by

$$\tilde{\mathcal{H}}_* = \tilde{H}_*(\tilde{\mathcal{C}}_*) \text{ and } \tilde{\mathcal{H}}_*^L = \tilde{H}_*(\tilde{\mathcal{C}}_*^L),$$

called the *reduced (Lipschitz) measure homologies*. They are also represented as

$$\begin{aligned} \tilde{\mathcal{H}}_*(X) &= \ker(\mathcal{H}_*(X) \rightarrow \mathcal{H}_*(\{*\})) \\ \tilde{\mathcal{H}}_*^L(Y) &= \ker(\mathcal{H}_*^L(Y) \rightarrow \mathcal{H}_*^L(\{*\})) \end{aligned}$$

for a topological space X and a metric space Y , where the maps between homologies are induced by the trivial maps $X \rightarrow \{*\}$ and $Y \rightarrow \{*\}$ to a one-point space. From the definition, the following trivially holds.

Lemma 4.8. *For a one-point space $\{*\}$, $\tilde{\mathcal{H}}_*(\{*\}) = 0$ and $\tilde{\mathcal{H}}_*^L(\{*\}) = 0$.*

The following statement is an expression of a statement proved in [H] and [Z] in other words.

Theorem 4.9 ([H], [Z]). *Let U be a subset of a topological space V which is contractible in V . Namely, there is a continuous map $h : U \times [0, 1] \rightarrow V$ such that h_0 is the inclusion $U \hookrightarrow V$ and h_1 is the constant map to some point in V . Then, the inclusion h_0 induces the zero map $\tilde{\mathcal{H}}_k(U) \rightarrow \tilde{\mathcal{H}}_k(V)$ for every $k \geq 0$.*

In particular, for a locally contractible topological space X , $\tilde{\mathcal{H}}_k : \mathcal{O}(X) \rightarrow \mathbf{Vect}$ is a locally trivial precosheaf for each $k \geq 0$.

In contrast to Theorem 4.9, we prove

Theorem 4.10. *Let $U \subset V$ be Lipschitz contractible in a metric space V in the following sense. There is a Lipschitz map $h : U \times [0, 1] \rightarrow V$ such that h_0 is the inclusion $U \hookrightarrow V$ and h_1 is the constant map from U to a point in V . Then, the inclusion h_0 induces the zero map $\tilde{\mathcal{H}}_k^L(U) \rightarrow \tilde{\mathcal{H}}_k^L(V)$ for every $k \geq 0$.*

In particular, if X is a weakly locally Lipschitz contractible metric space, then $\tilde{\mathcal{H}}_k^L : \mathbf{O}(X) \rightarrow \mathbf{Vect}$ is a locally trivial precosheaf for each $k \geq 0$.

To prove Theorem 4.10, we review an outline of the proof of Theorem 4.9. Let us recall the standard prism decomposition of a singular simplex. For a regular k -simplex Δ^k , the prism $\Delta^k \times I$ has a standard decomposition into $(k + 1)$ -simplices P_i ($i = 0, \dots, k$):

$$\Delta^k \times I = \bigcup_{i=0}^k P_i.$$

Here, we assume that all simplices P_i have positive orientation associated to the orientation of Δ^k . Let $\sigma \in C(\Delta^k, X)$ be a singular simplex in a topological space X . We define $\sigma \times \text{id}_I : \Delta^k \times I \rightarrow X \times I$ by $(\sigma \times \text{id}_I)(s, t) = (\sigma(s), t)$ for $s \in \Delta^k$ and $t \in I$. Setting $P_i \sigma := (\sigma \times \text{id}_I)|_{P_i}$, the prism decomposition of σ is given by the formal sum

$$P\sigma = \sum_{i=0}^k P_i \sigma$$

Its linear extension $P : C_k(X) \rightarrow C_{k+1}(X \times I)$ is a chain homotopy between $i_{0\#}$ and $i_{1\#}$, i.e., it satisfies

$$(4.1) \quad \partial P - P\partial = i_{0\#} - i_{1\#}$$

where $C_*(\cdot)$ denotes the real singular chain complex. Here, the map $i_t : X \rightarrow X \times I$ is given by $i_t(x) = (x, t)$ for $t \in I$.

For each $i = 0, \dots, k$, the map $P_i : C(\Delta^k, X) \rightarrow C(\Delta^{k+1}, X \times I)$ is continuous in the compact-open topology. So, it induces a map $P_{i\#} : \mathcal{C}_k(X) \rightarrow \mathcal{C}_{k+1}(X \times I)$. The sum $P_{\#} = \sum_{i=0}^k P_{i\#}$ satisfies

$$(4.2) \quad \partial P_{\#} - P_{\#}\partial = i_{0\#} - i_{1\#}.$$

The relation (4.2) can be checked by a similar way to check the relation (4.1). This implies Theorem 4.9, due to Lemma 4.8.

Proof of Theorem 4.10. Let X be a metric space. If $\sigma : \Delta^k \rightarrow X$ is Lipschitz, then $P_i \sigma$ is obviously Lipschitz. Further, the map

$$P_i : \text{Lip}(\Delta^k, X) \rightarrow \text{Lip}(\Delta^{k+1}, X \times I)$$

is continuous in the our topology, due to (4) and (3) of Theorem 3.1. Then, the sum

$$P_{\#} = \sum_{i=0}^k P_{i\#} : \mathcal{C}_k^L(X) \rightarrow \mathcal{C}_{k+1}^L(X \times I)$$

is a chain homotopy between $i_{0\#}$ and $i_{1\#}$. This observation and Lemma 4.8 imply the conclusion of Theorem 4.10. \square

An augmentation $\tilde{\partial}_0$ of the current chain complex $\mathbf{N}_*^c(X)$ was considered in [RS] and was defined by

$$\tilde{\partial}_0 : \mathbf{N}_0^c(X) \ni T \mapsto T(1) \in \mathbb{R}.$$

Actually, it satisfies $\tilde{\partial}_0 \partial_1 T = 0$ for $T \in \mathbf{N}_1^c(X)$.

Lemma 4.11. *Let X be a metric space. The following diagram*

$$\begin{array}{ccc} \mathcal{C}_0^L(X) & \xrightarrow{\mu \mapsto T^\mu} & \mathbf{N}_0^c(X) \\ & \searrow \tilde{\partial}_0 & \swarrow \tilde{\partial}_0 \\ & \mathbb{R} & \end{array}$$

commutes.

Proof. Let $\mu \in \mathcal{C}_0^L(X)$. Then, we have

$$\tilde{\partial}_0 T^\mu = T^\mu(1) = \mu(X) = \tilde{\partial}_0 \mu.$$

This completes the proof. \square

We already know the following

Theorem 4.12 ([Mi]). *If X is a weakly locally Lipschitz contractible metric space, then the precosheaf \mathbf{H}_k on X is locally trivial for every $k \geq 1$.*

By summarizing above and using general cosheaf theory, we obtain

Proof of Theorem 1.8. This follows from Lemmas 3.20 and 4.11, Theorems 4.12, 4.9, 4.10 and 4.1 and Corollary 4.6. \square

Let us prove Corollaries 1.4, 1.5, 1.6 and 1.13.

Proof of Corollary 1.4. This follows from Theorems 1.3 and 1.12. \square

Proof of Corollary 1.5. Let X be a metric space of Hausdorff dimension $< n$, for a nonnegative integer n . Then, due to [AK, Theorem 3.9], $\mathbf{N}_k(X) = 0$ for every integer $k \geq n$. Therefore, we have $\mathbf{H}_k(X) = 0$ for $k \geq n$. Now we assume that X is weakly locally Lipschitz contractible. By Theorem 1.3, we obtain $\mathcal{H}_k(X) = 0$ for all $k \geq n$. \square

Proof of Corollary 1.6. Let X be a WLLC metric space and Y a finite CW-complex. Suppose that there is a homotopy equivalence $h : X \rightarrow Y$. By Theorem 1.12, there is a commutative diagram consisting of isomorphisms:

$$\begin{array}{ccc} H_*(X) & \longrightarrow & \mathcal{H}_*(X) \\ h_* \downarrow & & \downarrow h_* \\ H_*(Y) & \longrightarrow & \mathcal{H}_*(Y). \end{array}$$

Due to Theorem 1.8, we have $\mathcal{H}_*(X) \cong \mathbf{H}_*(X)$. From the assumption, $\dim H_k(Y) < \infty$ for all $k \geq 0$. Thus, we obtain the conclusion. \square

Proof of Corollary 1.13. This follows a way similar to the proof of Corollary 1.6. \square

We say that (X, A) is a pair of weakly locally Lipschitz contractible metric spaces if so are both X and A , where A has the restricted metric from X .

Theorem 4.13. *On the category of all pairs of metric spaces and all locally Lipschitz maps, there are natural transformations $\mathcal{C}_* \leftarrow \mathcal{C}_*^L \rightarrow \mathbf{N}_*^c$. If they are restricted to the category of all pairs of weakly locally Lipschitz contractible metric spaces and all locally Lipschitz maps, then they induce isomorphisms $\mathcal{H}_* \leftarrow \mathcal{H}_*^L \rightarrow \mathbf{H}_*$ between homologies.*

Proof. This follows from Theorem 1.8 together with a standard argument of homology algebra (the five lemma and snake lemma). \square

5. A TOPOLOGY ON THE SPACE OF BOUNDED LIPSCHITZ MAPS

In this section, we provide a topology on the set of all bounded Lipschitz maps between general metric spaces which satisfies the properties as in Theorem 3.1 and additional properties.

5.1. Banach space target. In this subsection, we equip a topology on the set of all bounded Lipschitz maps from a metric space to a Banach space.

Let B be a Banach space over real numbers and X a metric space. Let $\text{Lip}_b(X, B)$ be the set of all bounded Lipschitz maps from X to B . Then, it is a real vector space associated to the standard addition and scalar multiplication operators. A topology on it is induced by the supremum norm $\|\cdot\|_\infty$ and the Lipschitz constant semi-norm $\text{Lip}(\cdot)$. The set $\text{Lip}_b(X, B)$ equipped with this topology is denoted by $\text{Lip}_b^{\text{BT}}(X, B)$. Obviously, it has a topology of a Banach space structure. This topology has the following fundamental properties.

Proposition 5.1. *Let X and X' be metric spaces and B and B' Banach spaces. Then, the following holds.*

- (a) Let $f_j, f \in \text{Lip}_b(X, B)$ with $j \in \mathbb{N}$. Then, f_j converges to f in the topology of $\text{Lip}_b^{\text{BT}}(X, B)$ if and only if $\|f_j - f\|_\infty \rightarrow 0$ and $\text{Lip}(f_j - f) \rightarrow 0$ as $j \rightarrow \infty$;
- (b) Let $\phi : B \rightarrow B'$ be a continuous linear map. Then, the map $\phi_\# : \text{Lip}_b^{\text{BT}}(X, B) \rightarrow \text{Lip}_b^{\text{BT}}(X, B')$ defined by the composition $\phi_\#(f) = \phi \circ f$, is a continuous linear map. In addition, if ϕ is injective and its image $\phi(B)$ is closed in B' , then $\phi_\#$ is a topological embedding.
- (c) Let $\psi : X \rightarrow X'$ be a Lipschitz map. Then, the composition $\psi^\# : \text{Lip}_b^{\text{BT}}(X', B) \rightarrow \text{Lip}_b^{\text{BT}}(X, B); f \mapsto f \circ \psi$ is a continuous linear map.
- (d) The map

$$\text{Lip}_b^{\text{BT}}(X, B) \times \text{Lip}_b^{\text{BT}}(X', B') \rightarrow \text{Lip}_b^{\text{BT}}(X \times X', B \times B')$$
 defined by $(f, g) \mapsto f \times g$ is a continuous linear map. Here, $f \times g$ is given by $(f \times g)(x, x') = (f(x), g(x'))$ for $x \in X$ and $x' \in X'$.
- (e) Let $\{*\}$ be a single-point set. Then, the map $\text{Lip}_b^{\text{BT}}(\{*\}, B) \ni f \mapsto f(*) \in B$ is a linear homeomorphism.

Proof. The property (d) follows from (b) and (c). The properties (a), (b), (c) and (e) are easily proved. We just demonstrate how (b) is proved. Let $\phi : B \rightarrow B'$ be a continuous linear map between Banach spaces B and B' , and X a metric space. It is trivial that $\phi_\#$ is linear. For any $f \in \text{Lip}_b(X, B)$, we have

$$\|\phi \circ f\|_\infty \leq \|\phi\|_{\text{op}} \|f\|_\infty \text{ and } \text{Lip}(\phi \circ f) \leq \|\phi\|_{\text{op}} \text{Lip}(f),$$

where $\|\phi\|_{\text{op}}$ is the operator norm of ϕ . Hence, $\phi_\#$ is continuous. The second statement of (b) follows from the inverse mapping theorem. \square

5.2. Double dual of metric spaces. Every metric space admits an isometric embedding into a Banach space. Several such constructions are known. Among them, we choose the following way. Let X be a metric space and $x_0 \in X$. Let $\text{Lip}_{x_0}(X)$ denote the Banach space of all real-valued Lipschitz functions on X vanishing at x_0 , equipped with the Lipschitz constant norm. We denote by $X_{x_0}^{**}$ the continuous dual of it. Then, a map $\delta : X \rightarrow X_{x_0}^{**}$ defined by

$$(5.1) \quad \delta_x(f) = f(x)$$

for all $x \in X$ and $f \in \text{Lip}_{x_0}(X)$, can be easily checked to be an isometric embedding. Obviously, δ_{x_0} becomes the zero vector in $X_{x_0}^{**}$. In this subsection, we observe fundamental properties of this construction. In particular, we see that the construction $(X, x_0) \mapsto X_{x_0}^{**}$ satisfies a covariant functorial property (Proposition 5.4).

Remark 5.2. The closed linear span of $\{\delta_x \in X_{x_0}^{**} \mid x \in X\}$ is called the (Lipschitz) free Banach space or the Arens-Eells space associated

to (X, x_0) (see e.g. [W]). One can employ it instead of $X_{x_0}^{**}$ after here, to achieve our purpose in §5.3.

For a Lipschitz map $\phi : X \rightarrow Y$ between metric spaces and $x_0 \in X$, we define

$$\phi_{\#} : X_{x_0}^{**} \rightarrow Y_{\phi(x_0)}^{**}$$

by

$$(\phi_{\#}\mu)(f) = \mu(f \circ \phi)$$

for all $\mu \in X_{x_0}^{**}$ and $f \in \text{Lip}_{\phi(x_0)}(Y)$.

Lemma 5.3. *For a Lipschitz map $\phi : X \rightarrow Y$ and $x_0 \in X$,*

$$\phi_{\#} \circ \delta = \delta \circ \phi$$

holds.

Proof. Let $x \in X$ and $f \in \text{Lip}_{x_0}(X)$. Then, $\phi_{\#}\delta_x(f) = \delta_x(f \circ \phi) = f(\phi(x)) = \delta_{\phi(x)}(f)$. Namely, $\phi_{\#} \circ \delta = \delta \circ \phi$ holds. \square

Proposition 5.4. *Let $\phi : X \rightarrow Y$ and $x_0 \in X$ as above. Then, $\phi_{\#} : X_{x_0}^{**} \rightarrow Y_{\phi(x_0)}^{**}$ is a bounded linear operator with operator norm $\text{Lip}(\phi)$.*

Proof. It is trivial that $\phi_{\#}$ is linear. For $f \in \text{Lip}_{\phi(x_0)}(Y)$,

$$|\phi_{\#}\mu(f)| = |\mu(f \circ \phi)| \leq \|\mu\| \text{Lip}(f) \text{Lip}(\phi).$$

Hence, $\|\phi_{\#}\mu\| \leq \|\mu\| \text{Lip}(\phi)$. It implies $\|\phi_{\#}\| \leq \text{Lip}(\phi)$. By Lemma 5.3, for every $x \neq y \in X$, we have

$$\frac{\|\phi_{\#}\delta_x - \phi_{\#}\delta_y\|}{\|\delta_x - \delta_y\|} = \frac{d(\phi(x), \phi(y))}{d(x, y)}.$$

Since this value can be taken to be arbitrary close to $\text{Lip}(\phi)$, we obtain $\|\phi_{\#}\| = \text{Lip}(\phi)$. \square

Corollary 5.5. *If $\phi : X \rightarrow Y$ is a bi-Lipschitz homeomorphism and $x_0 \in X$, then $\phi_{\#} : X_{x_0}^{**} \rightarrow Y_{\phi(x_0)}^{**}$ is a linear homeomorphism.*

Let us recall the statement of McShane-Whitney's Lipschitz extension theorem: Let X be a subset of a metric space Y . For any Lipschitz function $f : X \rightarrow \mathbb{R}$, there is a Lipschitz function $g : Y \rightarrow \mathbb{R}$ such that $g|_X = f$ and $\text{Lip}(g) = \text{Lip}(f)$.

Lemma 5.6. *If X is a subset of a metric space Y , then the inclusion $i : X \rightarrow Y$ induces an isometric linear embedding $i_{\#} : X_{x_0}^{**} \rightarrow Y_{x_0}^{**}$ for every $x_0 \in X$.*

Proof. Let $x_0 \in X$ be fixed. The inclusion $i : X \hookrightarrow Y$ induces a linear map $i^{\#} : \text{Lip}_{x_0}(Y) \rightarrow \text{Lip}_{x_0}(X)$ given by $f \mapsto f \circ i = f|_X$ for $f \in \text{Lip}_{x_0}(Y)$. Since $\text{Lip}(f|_X) \leq \text{Lip}(f)$ for all $f \in \text{Lip}_{x_0}(Y)$, the operator norm of $i^{\#}$ is not greater than 1. Due to the McShane-Whitney Lipschitz extension theorem, the map $i^{\#}$ is surjective. Dually,

the bounded linear operator $i_{\#} : X_{x_0}^{**} \rightarrow Y_{x_0}^{**}$ is injective and its operator norm ≤ 1 . We prove that $\|i_{\#}\mu\| = \|\mu\|$ for every $\mu \in X_{x_0}^{**}$. We assume that there exist $\epsilon > 0$ and $\mu \in X_{x_0}^{**}$ such that $\|i_{\#}\mu\| \leq (1 - \epsilon)\|\mu\|$. We may assume that $\|\mu\| = 1$. Hence, we have

$$|\mu(f|_X)| \leq (1 - \epsilon)\text{Lip}(f)$$

for every $f \in \text{Lip}_{x_0}(Y)$. Again, due to the McShane-Whitney extension theorem, for every $g \in \text{Lip}_{x_0}(X)$, there is $f \in \text{Lip}_{x_0}(Y)$ such that $f|_X = g$ and $\text{Lip}(f) = \text{Lip}(g)$. This yields,

$$|\mu(g)| \leq (1 - \epsilon)\text{Lip}(g)$$

for every $g \in \text{Lip}_{x_0}(X)$. It implies $\|\mu\| \leq 1 - \epsilon$ which contradicts to the assumption $\|\mu\| = 1$. Therefore, $\|i_{\#}\mu\| = \|\mu\|$ holds for every μ . \square

Corollary 5.7. *If $\phi : X \rightarrow Y$ be a bi-Lipschitz embedding, then $\phi_{\#} : X_{x_0}^{**} \rightarrow Y_{\phi(x_0)}^{**}$ is an injective bounded linear map having closed image.*

Proof. This follows from Lemma 5.6 and Corollary 5.5. \square

For $x_0, x_1 \in X$, a canonical isometric isomorphism

$$\text{Lip}_{x_0}(X) \rightarrow \text{Lip}_{x_1}(X)$$

is defined by $f \mapsto f - f(x_1)$. It implies an isometric isomorphism

$$(5.2) \quad X_{x_1}^{**} \rightarrow X_{x_0}^{**}.$$

Namely, for $\mu \in X_{x_1}^{**}$, the map (5.2) assigns an element $\mu' \in X_{x_0}^{**}$ defined by

$$\mu'(f) = \mu(f - f(x_1))$$

for all $f \in \text{Lip}_{x_0}(X)$.

Lemma 5.8. *Let X and Y be metric spaces with $x_0 \in X$ and $y_0 \in Y$. Then, a map*

$$\Phi : X_{x_0}^{**} \times Y_{y_0}^{**} \rightarrow (X \times Y)_{(x_0, y_0)}^{**}$$

given by

$$\Phi(\mu, \nu)(h) = \mu(h(\cdot, y_0)) + \nu(h(x_0, \cdot))$$

*for $(\mu, \nu) \in X_{x_0}^{**} \times Y_{y_0}^{**}$ and $h \in \text{Lip}_{(x_0, y_0)}(X \times Y)$, is a continuous linear map.*

Proof. Let $(\mu, \nu) \in X_{x_0}^{**} \times Y_{y_0}^{**}$ and $h \in \text{Lip}_{(x_0, y_0)}(X \times Y)$. Then, we have

$$\begin{aligned} |\Phi(\mu, \nu)(h)| &\leq |\mu(h(\cdot, y_0))| + |\nu(h(x_0, \cdot))| \\ &\leq \text{Lip}(h)\{\|\mu\| + \|\nu\|\}. \end{aligned}$$

Therefore, Φ is a bounded linear map. \square

One can prove that the map Φ in Lemma 5.8 is injective and has the closed image.

When V is a Banach space, let us compare the space $V_0^{**} = \text{Lip}_0(V)^*$ with the usual continuous double dual V^{**} . Since the operator norm of a linear map is no other than its Lipschitz constant, the continuous dual V^* of V is contained in $\text{Lip}_0(V)$ as a closed subspace:

$$V^* \subset \text{Lip}_0(V).$$

Dually, we obtain a surjective bounded linear operator

$$(5.3) \quad r : V_0^{**} \twoheadrightarrow V^{**}$$

assigning the restriction $f|_{V^*}$ to V^* for each $f \in V_0^{**}$. The surjectivity follows from the Hahn-Banach theorem. Then, we obtain a canonical map

$$\bar{\delta} : V \rightarrow V^{**}$$

defined by

$$\bar{\delta}_x = \delta_x|_{V^*}$$

for $x \in V$. This map $\bar{\delta}$ is no other than the usual canonical isometric linear embedding of V into V^{**} defined by the evaluation.

5.3. Metric space target. Let A and X denote metric spaces. We equip $\text{Lip}_b(A, X)$ with a topology satisfying the desired properties stated in Theorem 3.1 and additional properties.

We fix $x_0 \in X$ and an isometric embedding $\delta : X \rightarrow X_{x_0}^{**}$ defined in (5.1). It implies an injection

$$\delta_{\#} : \text{Lip}_b(A, X) \rightarrow \text{Lip}_b(A, X_{x_0}^{**})$$

given by $\delta_{\#}f = \delta_f$ for all $f \in \text{Lip}_b(A, X)$. We endow $\text{Lip}_b(A, X)$ with the coarsest topology in which the map $\delta_{\#} : \text{Lip}_b(A, X) \rightarrow \text{Lip}_b^{\text{BT}}(A, X_{x_0}^{**})$ is continuous.

Lemma 5.9. *The topology on $\text{Lip}_b(A, X)$ given as above, is independent on the choice of base point x_0 .*

Proof. Let us fix $x_0, x_1 \in X$. We set maps

$$\delta^0 : X \rightarrow X_{x_0}^{**} \text{ and } \delta^1 : X \rightarrow X_{x_1}^{**}$$

given by the same maps $\delta^0 = \delta^1 = \delta$. Let $\Phi : X_{x_1}^{**} \rightarrow X_{x_0}^{**}$ denote a canonical map defined in (5.2). Then,

$$\Phi_{\#} : \text{Lip}_b^{\text{BT}}(A, X_{x_1}^{**}) \rightarrow \text{Lip}_b^{\text{BT}}(A, X_{x_0}^{**})$$

is homeomorphic due to Proposition 5.1 (b) and Corollary 5.5. For $f \in \text{Lip}_b(A, X)$ and $h \in \text{Lip}_{x_0}(X)$, we have

$$\begin{aligned} \Phi_{\#} \delta_{\#}^1 f(h) &= \Phi \circ \delta_f(h) = \delta_f(h - h(x_1)) = h \circ f(\cdot) - h(x_1) \text{ and,} \\ \delta_{\#}^0 f(h) &= \delta_f(h) = h \circ f(\cdot). \end{aligned}$$

Therefore, $\Phi_{\#}\delta_{\#}^1 - \delta_{\#}^0 : \text{Lip}_b(A, X) \rightarrow \text{Lip}_b^{\text{BT}}(A, X_{x_0}^{**})$ is a constant map. Since $\Phi_{\#}$ is homeomorphic, $\delta_{\#}^1$ is continuous if and only if so is $\delta_{\#}^0$. This completes the proof. \square

Let us denote by $\text{Lip}_b^{\text{MT}}(A, X)$ the space $\text{Lip}_b(A, X)$ with the topology induced by $\delta_{\#}$.

Proposition 5.10. *The space $\text{Lip}_b^{\text{MT}}(A, X)$ is metrizable. If a sequence f_j converges to f in $\text{Lip}_b^{\text{MT}}(A, X)$, then $\text{Lip}(f_j) \rightarrow \text{Lip}(f)$ and f_j converges to f uniformly as $j \rightarrow \infty$. In particular, the statements (0) and (1) of Theorem 3.1 hold.*

Proof. This follows from the definition of the topology. \square

Proposition 5.11. *The statement (3) of Theorem 3.1 holds. Namely, for $\phi : A \rightarrow A'$ a Lipschitz map between metric spaces, the map $\phi^{\#} : \text{Lip}_b^{\text{MT}}(A', X) \rightarrow \text{Lip}_b^{\text{MT}}(A, X)$ defined by $\phi^{\#}f = f \circ \phi$ is continuous.*

Proof. Let us fix $x_0 \in X$. The following diagram

$$\begin{array}{ccc} \text{Lip}_b^{\text{MT}}(A', X) & \xrightarrow{\delta_{\#}} & \text{Lip}_b^{\text{BT}}(A', X_{x_0}^{**}) \\ \phi^{\#} \downarrow & & \downarrow \phi^{\#} \\ \text{Lip}_b^{\text{MT}}(A, X) & \xrightarrow{\delta_{\#}} & \text{Lip}_b^{\text{BT}}(A, X_{x_0}^{**}) \end{array}$$

consisting of canonical maps, commutes. From Proposition 5.1 (c), we obtain the conclusion. \square

Proposition 5.12. *The statement (2) of Theorem 3.1 holds. Namely, for a Lipschitz map $\phi : X \rightarrow Y$ between metric spaces, the map $\phi_{\#} : \text{Lip}_b^{\text{MT}}(A, X) \rightarrow \text{Lip}_b^{\text{MT}}(A, Y); f \mapsto \phi \circ f$ is continuous. Further, if ϕ is a bi-Lipschitz embedding, then $\phi_{\#}$ is a topological embedding.*

Proof. The following diagram

$$\begin{array}{ccc} \text{Lip}_b^{\text{MT}}(A, X) & \xrightarrow{\delta_{\#}} & \text{Lip}_b^{\text{BT}}(A, X_{x_0}^{**}) \\ \phi_{\#} \downarrow & & \downarrow \phi_{\#\#} \\ \text{Lip}_b^{\text{MT}}(A, Y) & \xrightarrow{\delta_{\#}} & \text{Lip}_b^{\text{BT}}(A, Y_{\phi(x_0)}^{**}) \end{array}$$

consisting of canonical maps, commutes. By proposition 5.4, the map $\phi_{\#} : X_{x_0}^{**} \rightarrow Y_{\phi(x_0)}^{**}$ is a bounded linear map. Due to Proposition 5.1 (b), the map $\phi_{\#\#} : \text{Lip}_b^{\text{BT}}(A, X_{x_0}^{**}) \rightarrow \text{Lip}_b^{\text{BT}}(A, Y_{\phi(x_0)}^{**})$ is continuous. Therefore, $\phi_{\#} : \text{Lip}_b^{\text{MT}}(A, X) \rightarrow \text{Lip}_b^{\text{MT}}(A, Y)$ is continuous. The second statement follows from Proposition 5.1 (b) and Corollary 5.7. \square

Remark 5.13. To prove Proposition 5.12, it is important that the functor $(X, x_0) \mapsto X_{x_0}^{**}$ is covariant (Proposition 5.4) and that canonical isometric embeddings and this functor commutes (Lemma 5.3).

Proposition 5.14. *The statement (4) of Theorem 3.1 holds. Namely, for metric spaces A, B, X and Y , the canonical map*

$$\text{Lip}_b^{\text{MT}}(A, X) \times \text{Lip}_b^{\text{MT}}(B, Y) \rightarrow \text{Lip}_b^{\text{MT}}(A \times B, X \times Y)$$

is continuous.

Proof. Let us fix $x_0 \in X$ and $y_0 \in Y$. Let us consider the following commutative diagram

$$\begin{array}{ccc} \text{Lip}_b^{\text{MT}}(A, X) \times \text{Lip}_b^{\text{MT}}(B, Y) & \longrightarrow & \text{Lip}_b^{\text{MT}}(A \times B, X \times Y) \\ \downarrow & & \downarrow \\ \text{Lip}_b^{\text{BT}}(A, X_{x_0}^{**}) \times \text{Lip}_b^{\text{BT}}(B, Y_{y_0}^{**}) & & \\ \downarrow & & \downarrow \\ \text{Lip}_b^{\text{BT}}(A \times B, X_{x_0}^{**} \times Y_{y_0}^{**}) & \longrightarrow & \text{Lip}_b^{\text{BT}}(A \times B, (X \times Y)_{(x_0, y_0)}^{**}) \end{array}$$

consisting of canonical maps. Since the right downward arrow is a topological embedding, the top rightward arrow is continuous if and only if the composition of the left two downward arrows and the bottom rightward arrow is continuous. It follows from Lemma 5.8 and Proposition 5.1 (d). \square

Proposition 5.15. *The statement (5) of Theorem 3.1 holds. Namely, for a singleton set $\{*\}$ and a metric space X , the canonical map $X \rightarrow \text{Lip}_b^{\text{MT}}(\{*\}, X)$ is homeomorphic.*

Proof. This follows from Proposition 5.1 (e) and the definition of the topology. \square

When V is a Banach space and Z is a metric space, we compare topologies of $\text{Lip}_b^{\text{MT}}(Z, V)$ and $\text{Lip}_b^{\text{BT}}(Z, V)$.

Proposition 5.16. *Let V be a Banach space and Z a metric space. Then, the topologies on $\text{Lip}_b^{\text{MT}}(Z, V)$ and $\text{Lip}_b^{\text{BT}}(Z, V)$ coincide with each other.*

Proof. We first prove that the identity $\text{id} : \text{Lip}_b^{\text{MT}}(Z, V) \rightarrow \text{Lip}_b^{\text{BT}}(Z, V)$ is continuous. Let us consider the following commutative diagram

$$\begin{array}{ccc} \text{Lip}_b^{\text{MT}}(Z, V) & \xrightarrow{\text{id}} & \text{Lip}_b^{\text{BT}}(Z, V) \\ \delta_{\#} \downarrow & & \downarrow \bar{\delta}_{\#} \\ \text{Lip}_b^{\text{BT}}(Z, V_0^{**}) & \xrightarrow{r_{\#}} & \text{Lip}_b^{\text{BT}}(Z, V^{**}). \end{array}$$

Here, the bottom rightward arrow $r_{\#}$ is continuous, because it is induced by the bounded linear map $r : V_0^{**} \rightarrow V^{**}$ given in (5.3). Since $\bar{\delta} : V \rightarrow V^{**}$ is an isometric linear embedding, the induced map $\bar{\delta}_{\#}$ is a topological embedding, due to Proposition 5.1 (b). It follows from the

continuity of $r_{\#} \circ \delta_{\#}$ that $\text{id} : \text{Lip}_b^{\text{MT}}(Z, V) \rightarrow \text{Lip}_b^{\text{BT}}(Z, V)$ is continuous.

Every neighborhood at 0 in $\text{Lip}_b^{\text{MT}}(Z, V)$ is generated by sets of form

$$\delta_{\#}^{-1}(\{g \in \text{Lip}_b(Z, V_0^{**}) \mid \|g\|_{\infty} < \rho \text{ and } \text{Lip}(g) < \ell\})$$

for $\rho, \ell > 0$. Since δ is the isometric embedding, these sets are equal to

$$\{f \in \text{Lip}_b(Z, V) \mid \|f\|_{\infty} < \rho \text{ and } \text{Lip}(f) < \ell\}$$

which are also open neighborhoods of 0 in $\text{Lip}_b^{\text{BT}}(Z, V)$. Hence, the identity $\text{id} : \text{Lip}_b^{\text{BT}}(Z, V) \rightarrow \text{Lip}_b^{\text{MT}}(Z, V)$ is continuous at 0. Since $\text{Lip}_b^{\text{MT}}(Z, V)$ is a topological group by Proposition 3.9, the group homomorphism $\text{id} : \text{Lip}_b^{\text{BT}}(Z, V) \rightarrow \text{Lip}_b^{\text{MT}}(Z, V)$ is continuous on the whole set. \square

As a corollary to Propositions 5.16 and 5.12, we obtain

Corollary 5.17. *Let Z and X be metric space. Let $\phi : X \rightarrow V$ be a bi-Lipschitz embedding into a Banach space V . Then, the map $\phi_{\#} : \text{Lip}_b^{\text{MT}}(Z, X) \rightarrow \text{Lip}_b^{\text{BT}}(Z, V)$ is a topological embedding.*

Finally, we remark a relation between our topology on the space of Lipschitz maps and the C^1 -topology on the space of smooth maps, when a domain and a target are smooth compact Riemannian manifolds.

Let us denote by M and N compact smooth manifolds, where N has no boundary and M possibly has piecewise smooth boundary. The set of all C^1 -maps from M to N is denoted by $C^1(M, N)$. We fix Riemannian metrics on M and N , and regard them as metric spaces associated to the Riemannian metrics. Since every C^1 -map from M is Lipschitz, $C^1(M, N)$ is a subset of $\text{Lip}(M, N)$. The following gives a characterization of the C^1 -topology on $C^1(M, N)$ in terms of the topology of $\text{Lip}_b^{\text{MT}}(M, N)$:

Proposition 5.18. *Let M and N as above. The relative topology on $C^1(M, N)$ as a subset of $\text{Lip}_b^{\text{MT}}(M, N)$ coincides with the C^1 -topology on it.*

Proof. Let us take a Whitney smooth embedding $\Phi : N \rightarrow \mathbb{R}^K$ into a Euclidean space \mathbb{R}^K for a large $K \geq \dim N$. Then, Φ is also a bi-Lipschitz embedding, since N is compact. We have the following commutative diagram

$$\begin{array}{ccc} C^1(M, N) & \xrightarrow{\subset} & \text{Lip}_b^{\text{MT}}(M, N) \\ \Phi_{\#} \downarrow & & \downarrow \Phi_{\#} \\ C^1(M, \mathbb{R}^K) & \xrightarrow{\subset} & \text{Lip}_b^{\text{MT}}(M, \mathbb{R}^K). \end{array}$$

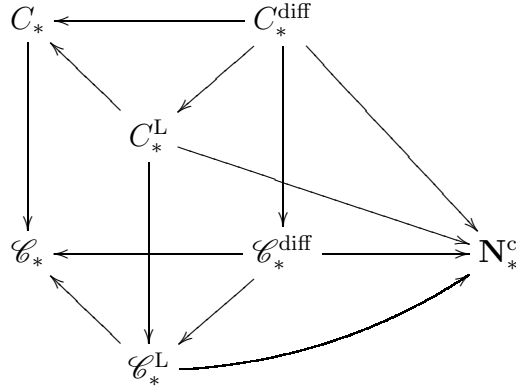
Since both two $\Phi_{\#}$ in this diagram are topological embeddings, if $C^1(M, \mathbb{R}^K) \subset \text{Lip}_b^{\text{MT}}(M, \mathbb{R}^K)$ is a topological embedding, then so is $C^1(M, N) \subset \text{Lip}_b^{\text{MT}}(M, N)$. Therefore, we may assume that $N = \mathbb{R}^K$

a Euclidean space. Then, $\text{Lip}_b^{\text{BT}}(M, \mathbb{R}^K)$ and $\text{Lip}_b^{\text{MT}}(M, \mathbb{R}^K)$ are same as topological spaces, due to Proposition 5.16. Let us take a sequence f_j and an element f in $C^1(M, \mathbb{R}^K)$. Since $\text{Lip}(g) = \|\nabla g\|_\infty$ for any $g \in C^1(M, \mathbb{R}^K)$, the sequence f_j converges to f in the topology of $\text{Lip}_b^{\text{MT}}(M, \mathbb{R}^K) = \text{Lip}_b^{\text{BT}}(M, \mathbb{R}^K)$ if and only if it converges to f in the C^1 -topology. This completes the proof. \square

By a similar way to the proof of Proposition 5.18, one can prove the following. Let M be a compact smooth Riemannian manifold possibly piecewise smooth boundary, and N an arbitrary smooth Riemannian manifold. Here, N is not necessarily compact. Then, the inclusion $C^1(M, N) \hookrightarrow \text{Lip}_b^{\text{MT}}(M, N)$ is continuous on each compact set. Due to it, a canonical chain map

$$\mathcal{C}_*^{\text{diff}} \rightarrow \mathcal{C}_*^{\text{L}}$$

is defined on the smooth category. Further, we obtain the commutative diagram which induce isomorphisms between homologies, on the smooth category:



6. A WLLC METRIC SPACE THAT IS NOT HOMOTOPIC TO ANY CW-COMPLEX

For terminologies and fundamental facts that will be stated in this section, we refer the book [S, Chapter 6]. This section is devoted to prove

Theorem 6.1. *There is a weakly locally Lipschitz contractible metric space X which has no homotopy type of an ANR.*

We recall that a Hausdorff space has the homotopy type of a simplicial complex if and only if it has the homotopy type of an ANR. Any CW-complex has the homotopy type of a simplicial complex. So, an X as in Theorem 6.1 has no homotopy type of a CW-complex.

Let Y be a Hausdorff space. For an open cover \mathcal{U} of Y , an open refinement \mathcal{V} of it is called an h -refinement if for every \mathcal{V} -close maps

$\phi, \psi : Z \rightarrow Y$ from an arbitrary Hausdorff space Z , there exists a \mathcal{U} -homotopy between them. Here, ϕ and ψ are said to be \mathcal{V} -close if for any $z \in Z$, there is $V \in \mathcal{V}$ such that $\{\phi(z), \psi(z)\} \subset V$, and a \mathcal{U} -homotopy between ϕ and ψ is a homotopy $h : Z \times [0, 1] \rightarrow Y$ between ϕ and ψ such that for every $z \in Z$, some $U \in \mathcal{U}$ exists with $\{h(z, t) \mid t \in [0, 1]\} \subset U$. Recall that every open cover of an arbitrary ANR has an h -refinement.

Lemma 6.2. *Let X be a topological space such that the cover $\{X\}$ has no h -refinement. Then, X is not homotopic to any ANR.*

Proof. Let us assume that X is homotopic to some ANR Y and fix a homotopy equivalence $f : X \rightarrow Y$. Since Y is an ANR, there exists an h -refinement \mathcal{U} of the cover $\{Y\}$. Let $f^{-1}\mathcal{U} := \{f^{-1}(U) \mid U \in \mathcal{U}\}$ be an open cover of X . Then, it can be easily checked that $f^{-1}\mathcal{U}$ is an h -refinement of $\{X\}$. It contradicts to the assumption. \square

Proof of Theorem 6.1. A Hausdorff space X which is locally contractible and satisfies the assumption of Lemma 6.2 was constructed, for instance in [S], as a subset with the relative topology in the product $I^{\mathbb{N}}$ of countably infinitely many intervals $I = [0, 1]$. By Lemma 6.2, X has no homotopy type of an ANR.

Let us consider the Hilbert cube Q defined by

$$Q = \{x = (x_k)_{k=1}^{\infty} \in \ell_2 \mid x_k \in [0, 2^{-k+1}] \text{ for } k \geq 1\}.$$

Let $\phi : I^{\mathbb{N}} \rightarrow Q$ be a standard homeomorphism given by

$$\phi(x) = (2^{-k+1}x_k)_{k=1}^{\infty}.$$

Then, $Y = \phi(X)$ has the same topological property as X . Further, the proof of the local contractibility of X given in [S] implies the weak local Lipschitz contractibility of Y , because local contractions in X were constructed explicitly by the translation of coordinates. This completes the proof. \square

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